POPL 2020

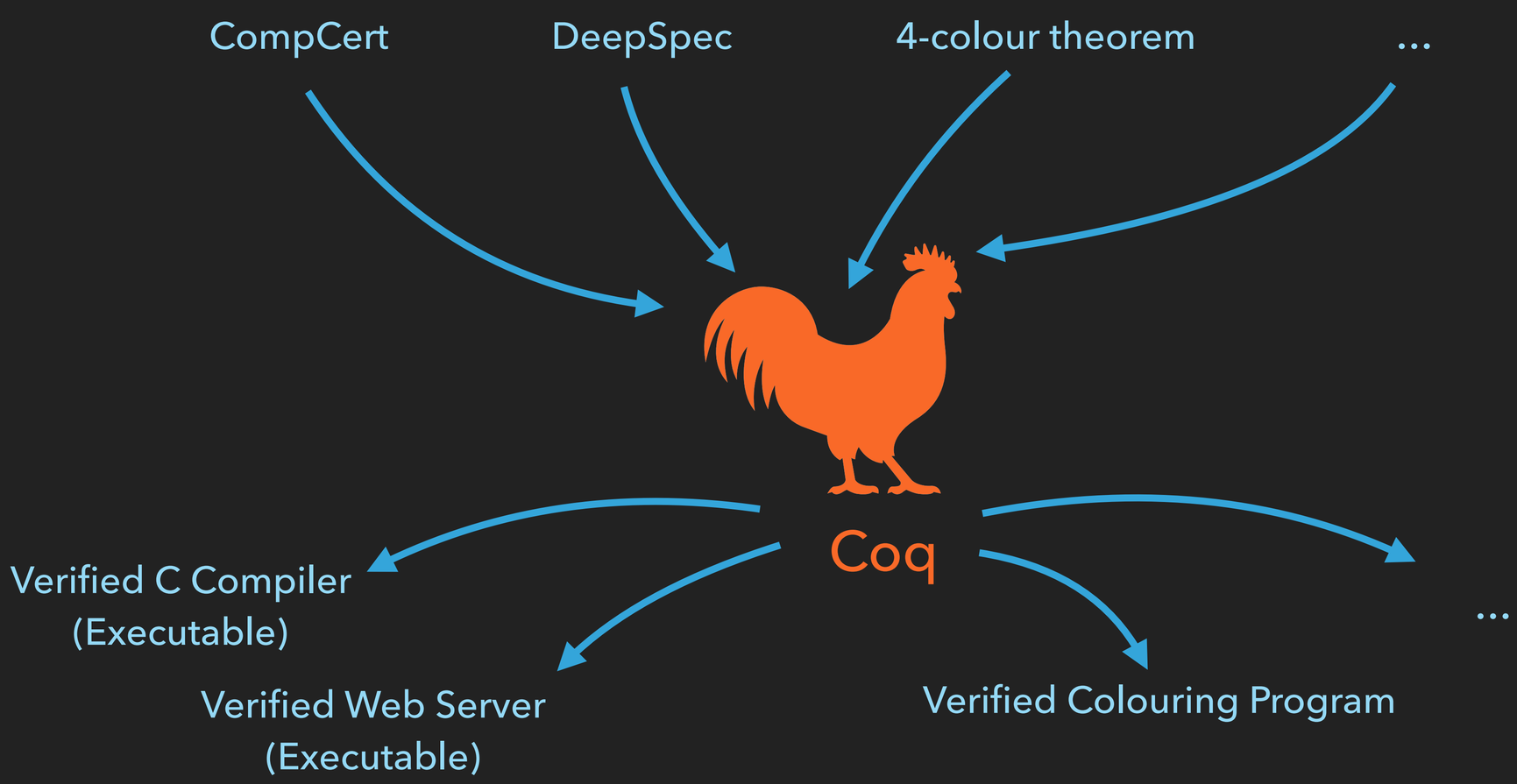
Coq Coq Correct!

Verification of Type Checking and Erasure for Coq, in Coq

Matthieu **Sozeau** Simon **Boulier** Yannick **Forster**

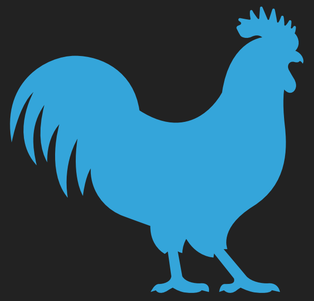
Nicolas **Tabareau** Théo **Winterhalter**

Motivation


CompCert
DeepSpec
4-colour theorem
...
Coq
Verified C Compiler
(Executable)
Verified Web Server
(Executable)
Verified Colouring Program
...

What do you trust?

Ideal Coq



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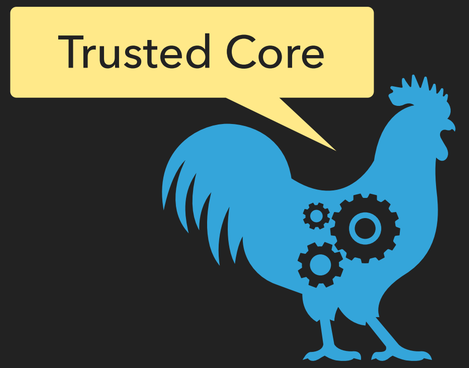

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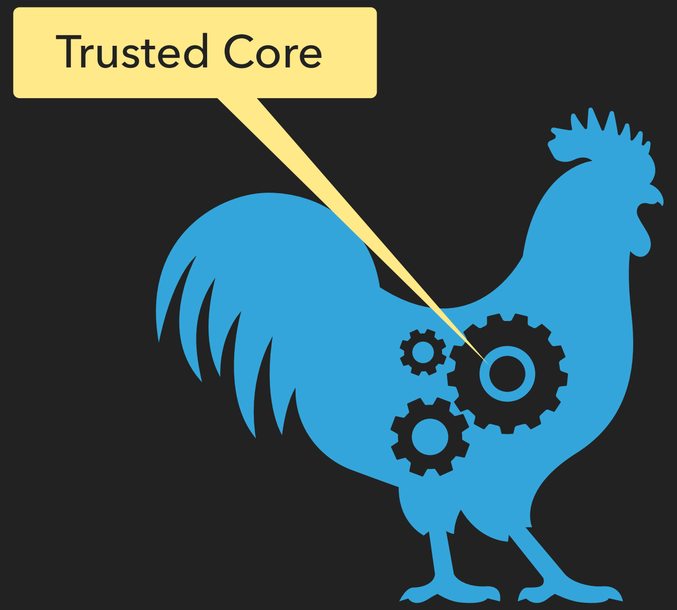
What do you trust?

Dependent Type Checker (18kLoC, 30+ years)

- Inductive Families w/ Termination Checker
- Universe Cumulativity and Polymorphism
- ML-style Module System
- KAM, VM and Native Conversion Checkers
- OCaml's Compiler and Runtime



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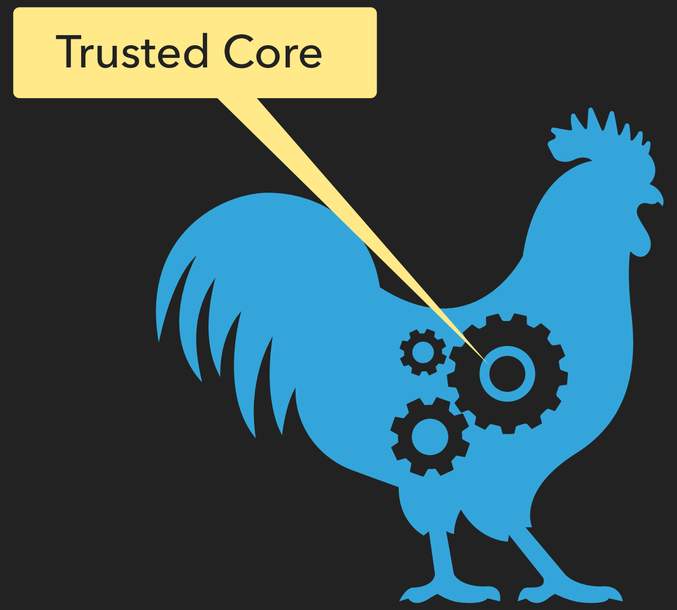
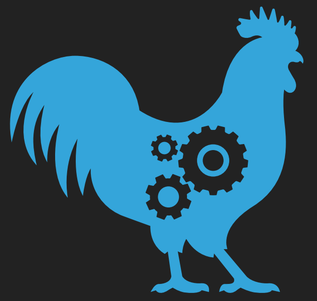


Implemented Coq

The Reality

Ideal Coq



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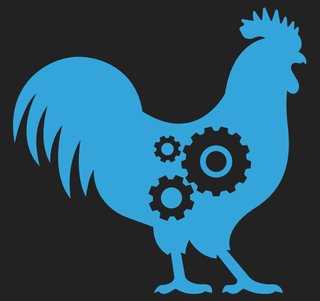

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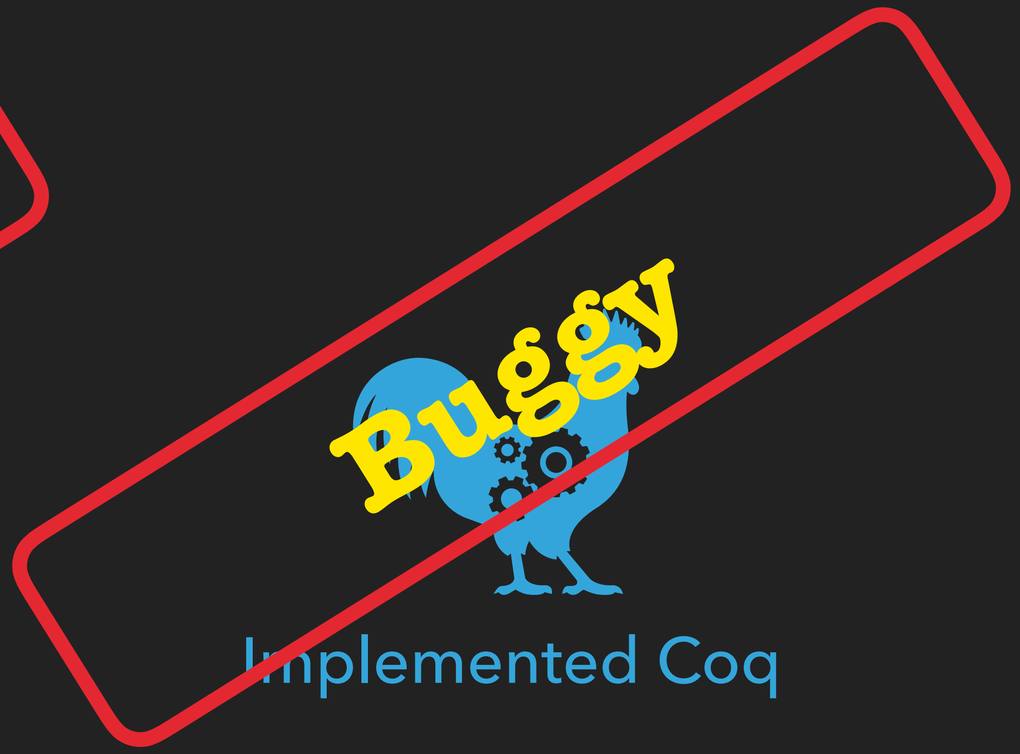


Implemented Coq

The Reality

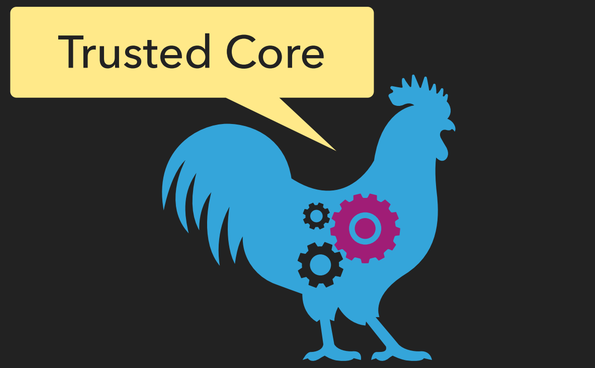
354 lines (314 sloc) | 16.7 KB

```
Preliminary compilation of critical bugs in stable releases of Coq
=================================================================
  WORK IN PROGRESS WITH SEVERAL OPEN QUESTIONS


To add: #7723 (vm_compute universe polymorphism), #7695 (modules an

Typing constructions

  component: "match"
  summary: substitution missing in the body of a let
  introduced: ?
  impacted released versions: V8.3-V8.3pl2, V8.4-V8.4pl4
  impacted development branches: none
  impacted coqchk versions: ?
  fixed in: master/trunk/v8.5 (e583a79b5, 22 Nov 2015, Herbelin), v
  found by: Herbelin
```



Implemented Coq

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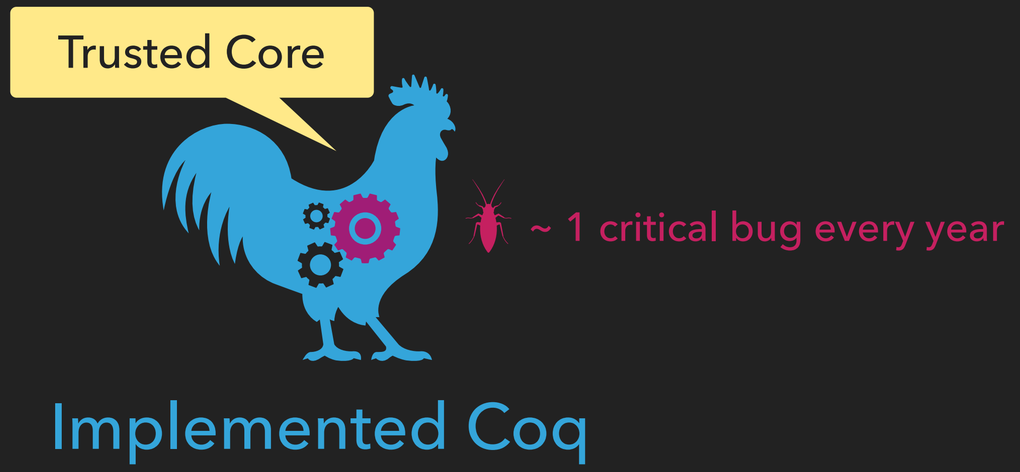
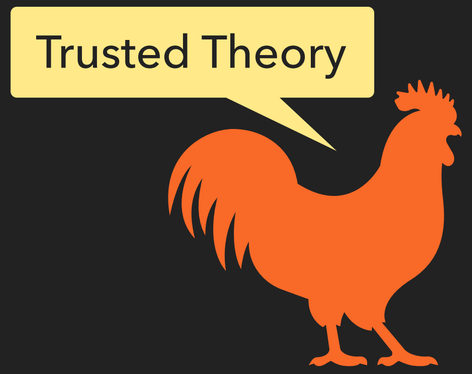


Our Goal: Improving Trust



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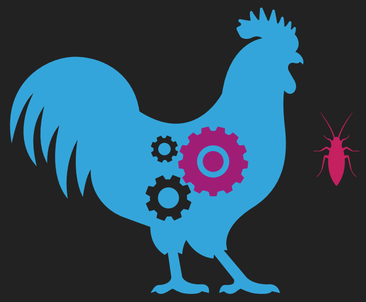

~ 1 critical bug every year

Implemented Coq

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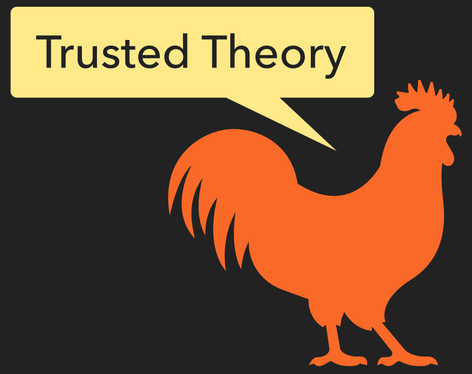

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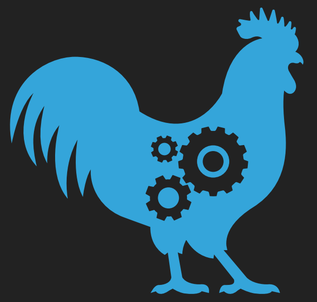
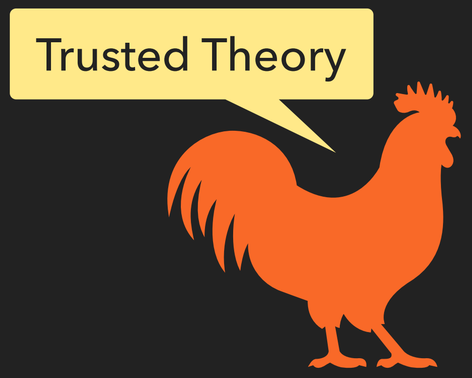
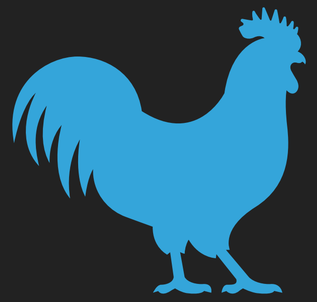


Implemented Coq

Coq in MetaCoq



Coq Spec: PCUIC



Verified Coq

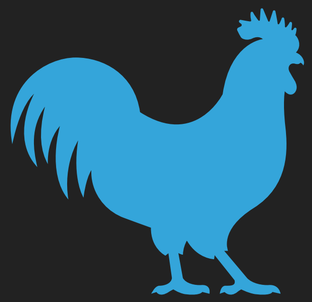
in

MetaCoq

Formalization of Coq in Coq

JAR'20

in



Implemented Coq

What we have...

```
fix vrev {A : Type@{i}} {n m : nat} (v : vec A n) (acc : vec A m) :=
  match v in vec _ n return vec A (n + m) with
  | vnil          ⇒ acc
  | vcons a n v' ⇒
      let idx := S n + m in
      coerce (vec A) idx (e : n + S m = idx) (vrev v' (vcons a m acc))
  end.
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```

```
vrev_term : term :=
tFix [{|
  dname := nNamed "vrev" ;
  dtype := tProd (nNamed « A") (tSort (Universe.make'' (Level.Level "Top.160", false) []))
    (tProd (nNamed "n") (tInd {| inductive_mind := "Coq.Init.Datatypes.nat";
      inductive_ind := 0 |} [])
    (tProd (nNamed "m") (tInd {| ...
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```

...and what we don't

```
(fun x ⇒ f x) ≡ f   (x ∉ f)
```

η-conversion

```
list nat : Set
list Type@{i} : Type@{i}
```

« template » polymorphism

```
Module M <: S. Definition t := nat. End M.
```

module system

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module system

no existential or named variables

Specification

Example: Reduction

$$\frac{(x : T := t) \in \Gamma}{\Gamma \vdash x \to t}$$

$$\frac{}{\Gamma \vdash \texttt{let}\ x : T := t\ \texttt{in}\ b \to b'[x := t]}$$

$$\frac{\Gamma \vdash t \to t' \qquad \Gamma, x : T := t \vdash b \to b'}{\Gamma \vdash \texttt{let}\ x : T := t\ \texttt{in}\ b \to \texttt{let}\ x : T := t'\ \texttt{in}\ b'}$$

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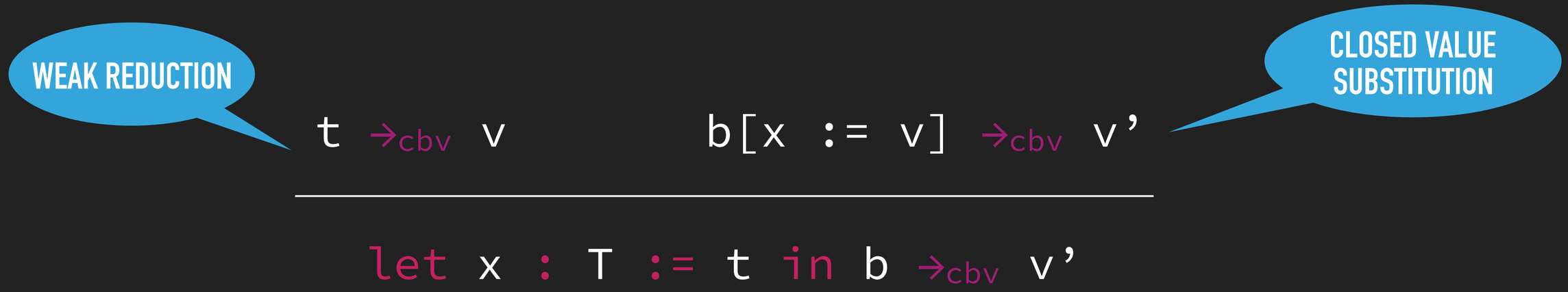
Specification

Example: Call-by-Value Evaluation



$$\_ \to_{cbv} \_ \quad \subseteq \quad \varepsilon \vdash \_ \to \_$$

Meta-Theory

Proven Properties

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- ***Structural Properties***: substitution, local and global weakening, instantiation by universes

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Proven Properties

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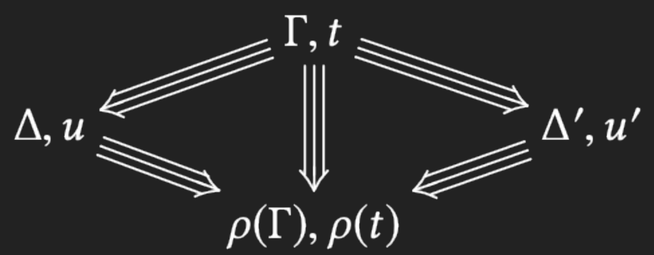
- ***Confluence***

  à la Tait-Martin-Löf/Takahashi



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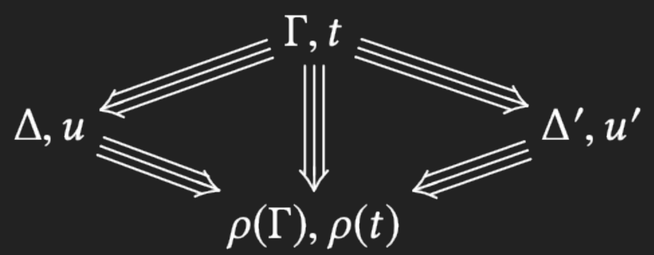

- Inversion Principles

Trusted Theory Base

Assumptions

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- The specifications of typing, reduction and cumulativity
  ~ 500 LoC from scratch

Trusted Theory Base

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- The specifications of typing, reduction and cumulativity
  ~ 500 LoC from scratch
- Strict Positivity & Guard Conditions
  Oracles: `check_fix : fixpoint -> bool`
- *Subject Reduction & Principality*
- Strong Normalization

Verifying Type-Checking

Reduction

Reduction

Input u

Reduction

Input u

Output v

Reduction

Input u

Output v

u → v

Reduction

Input `u : term`

Output `v` `u → v`

Reduction

Input `u : term`

Output `v : term`

`u → v`

Reduction

Input `u : term`

Output `v : term` `u → v : Prop`

Reduction

Input `u : term`

Output `v : term` `u → v : Prop`

```
weak_head_reduce : ∀ (u : term) → ∑ (v : term), u → v
```

Reduction

Reduction

Reduction

Reduction

Reduction

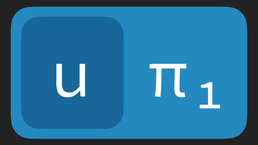
$t \longrightarrow \lambda x.x$

Reduction

$\lambda x.x \; u \; \pi_1$

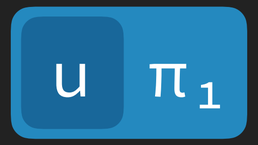
$t \longrightarrow \lambda x.x$

Reduction



$t \longrightarrow \lambda x.x$

Reduction



$$t\ u \longrightarrow (\lambda x.x)\ u \longrightarrow u$$

Reduction


```
match m with
| 0   ⇒ a
| S n ⇒ f n
end
```

Reduction


```
match m with
| 0     ⇒ a
| S n ⇒ f n
end
```

Reduction


```
match S p with
| 0     ⇒ a
| S n ⇒ f n
end
```

Reduction

Conversion

Specification

$$\frac{u \to w \qquad w \equiv v}{u \equiv v}$$

$$\frac{u \equiv_\alpha v}{u \equiv v}$$

$$\frac{u \equiv w \qquad v \to w}{u \equiv v}$$

Conversion

Algorithm

Conversion

Algorithm

Input `u : A`

Conversion

Algorithm

Input `u : A` `v : B`

Conversion

Algorithm

Input `u : A` `v : B`

Output `u ≡ v + error`

Conversion

Algorithm

Input `u : A` `v : B`

Output `u ≡ v + error`

```
isconv : ∀ (u v : term) → welltyped u → welltyped v → conv u v + error
```

Conversion

Algorithm

`u : A`

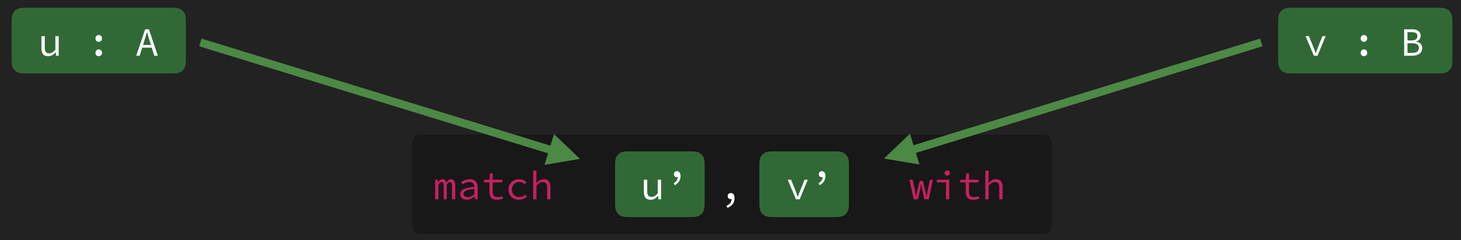
`v : B`

Conversion

Algorithm

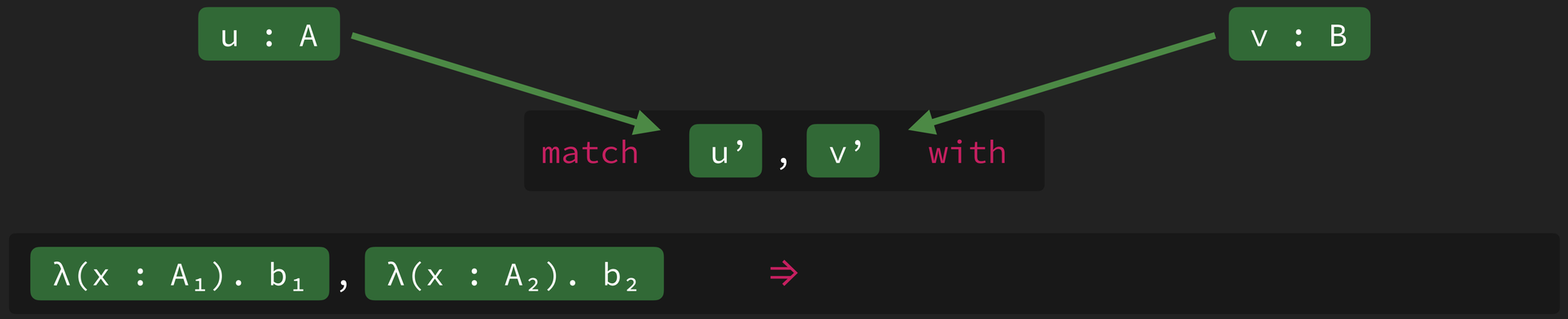
Conversion

Algorithm



Conversion

Algorithm



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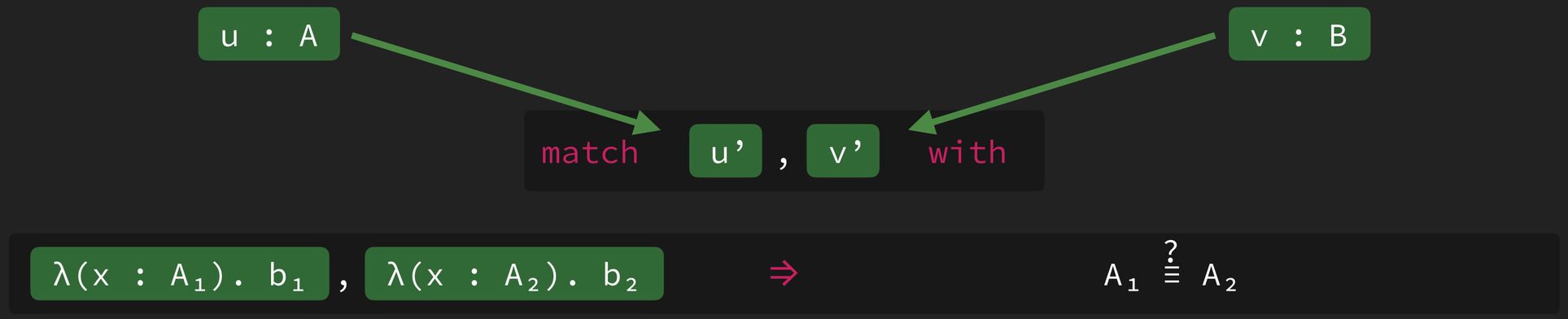

Conversion

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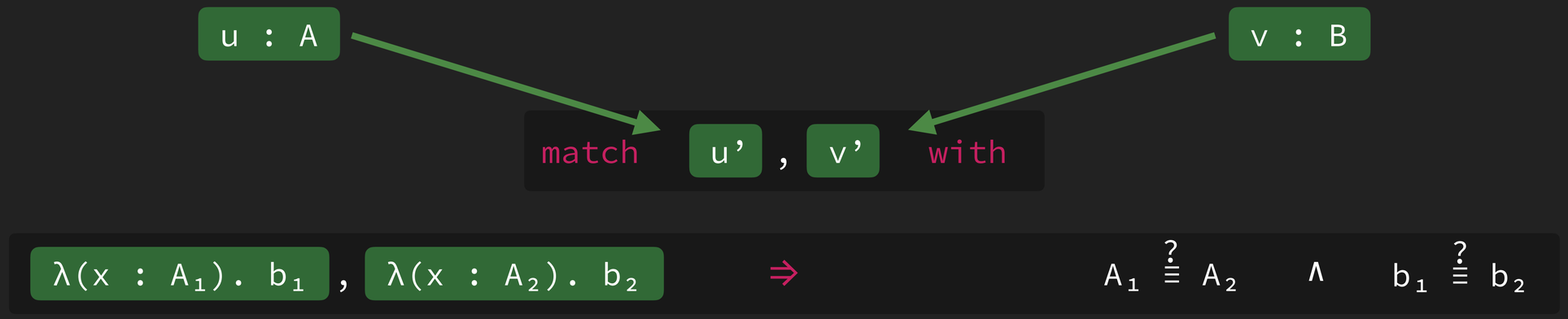

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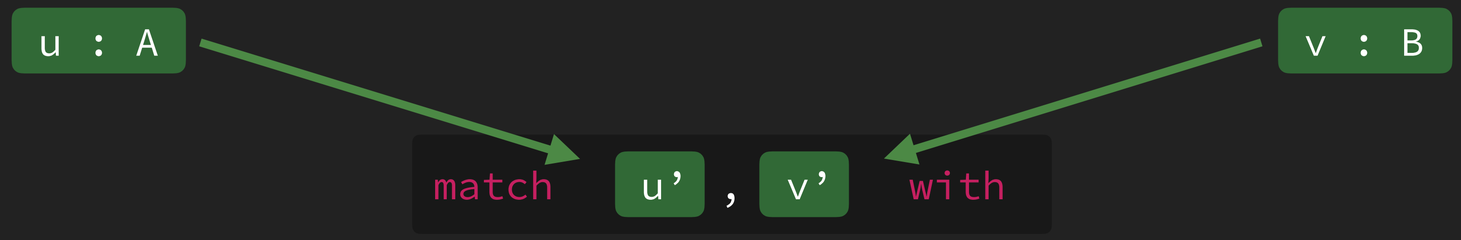

$\lambda(x : A_1).\ b_1$, $\lambda(x : A_2).\ b_2$ $\Rightarrow$ $A_1 \stackrel{?}{\equiv} A_2 \ \wedge \ b_1 \stackrel{?}{\equiv} b_2$

$\Pi(x : A_1).\ B_1$, $\Pi(x : A_2).\ B_2$ $\Rightarrow$ $A_1 \stackrel{?}{\equiv} A_2 \ \wedge \ B_1 \stackrel{?}{\equiv} B_2$

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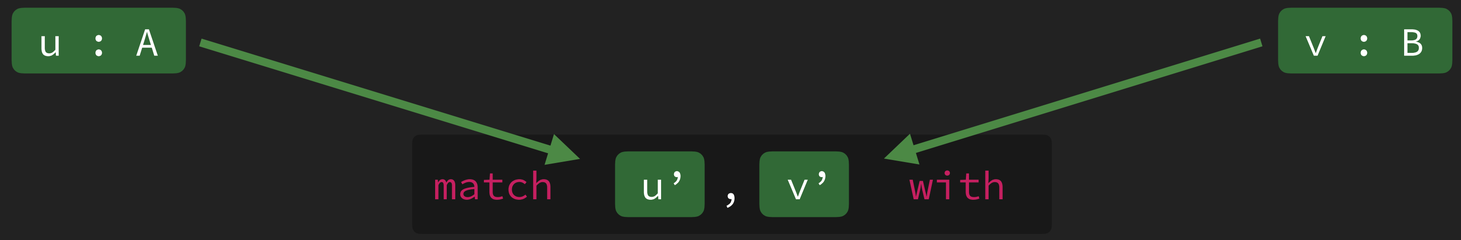


$\lambda(x : A_1).\ b_1$, $\lambda(x : A_2).\ b_2$ $\Rightarrow$ $A_1 \stackrel{?}{\equiv} A_2 \ \wedge \ b_1 \stackrel{?}{\equiv} b_2$

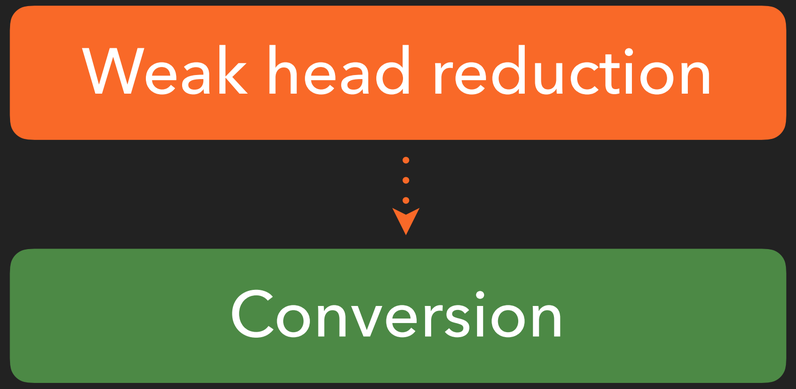
$\Pi(x : A_1).\ B_1$, $\Pi(x : A_2).\ B_2$ $\Rightarrow$ $A_1 \stackrel{?}{\equiv} A_2 \ \wedge \ B_1 \stackrel{?}{\equiv} B_2$

Type Checking

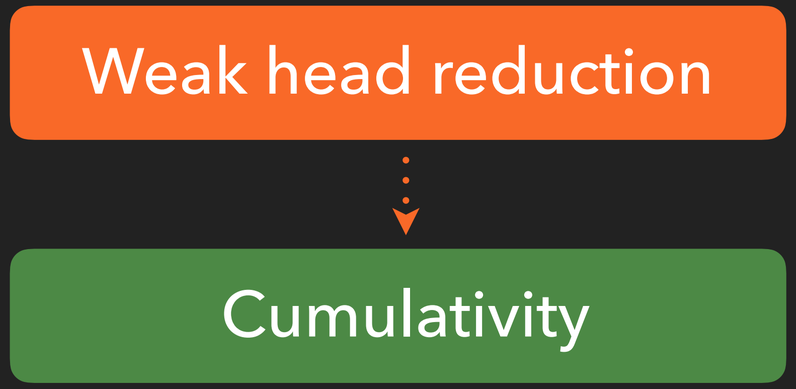
Type Checking

Weak head reduction

Type Checking

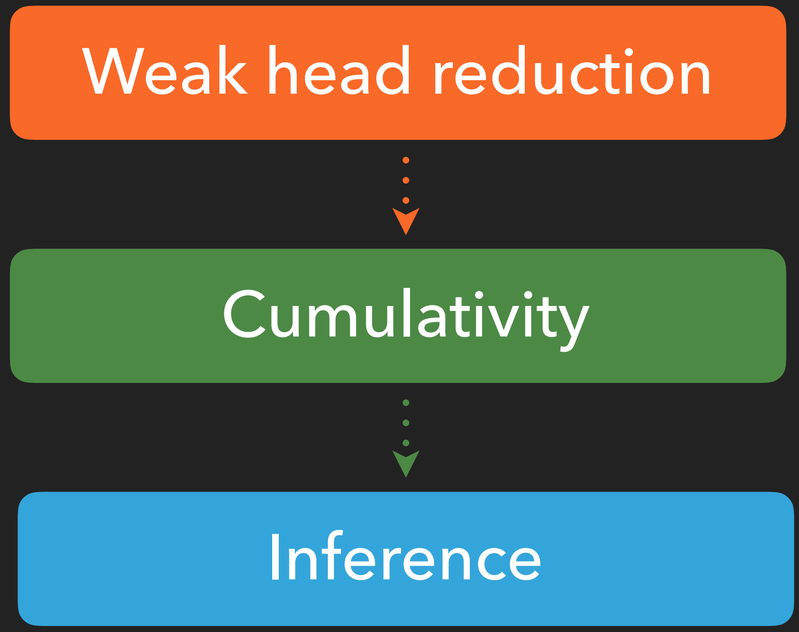


Type Checking



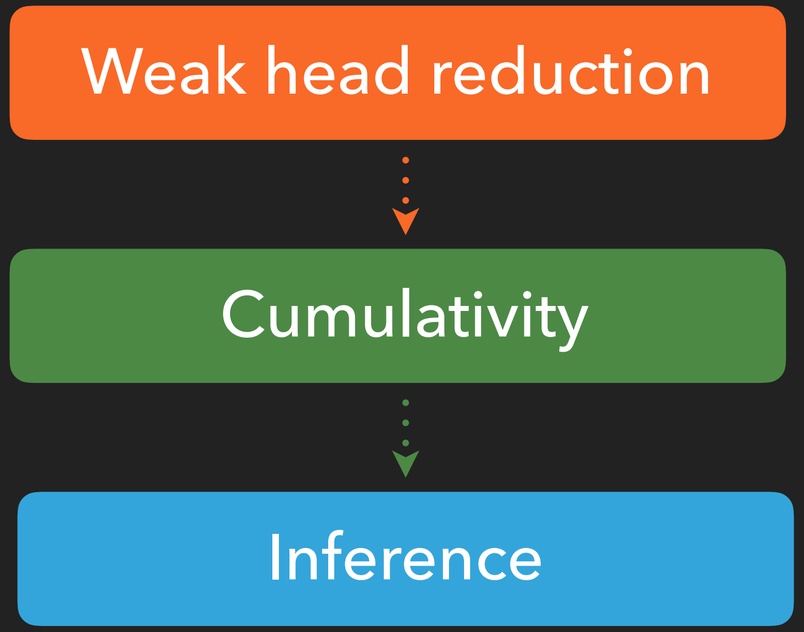
(Bellman-Ford)

Type Checking

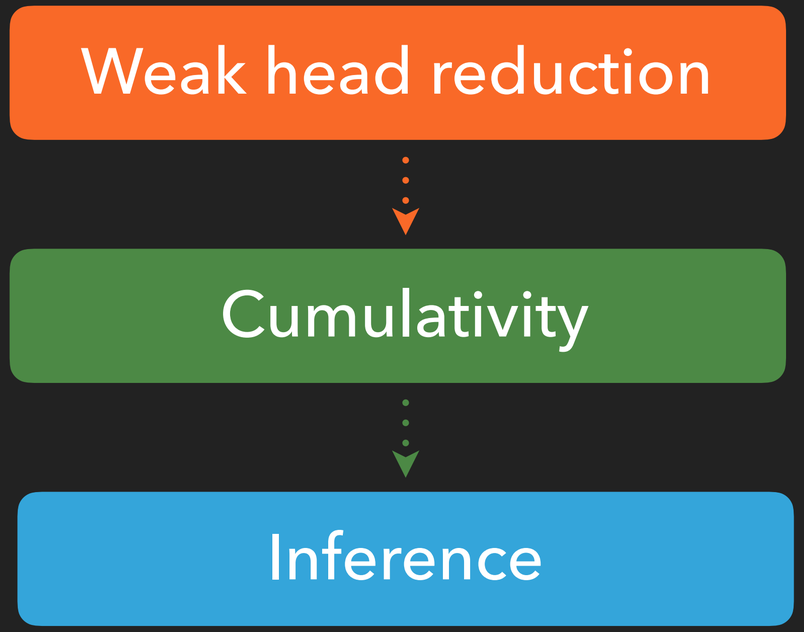


(Bellman-Ford)

Type Checking

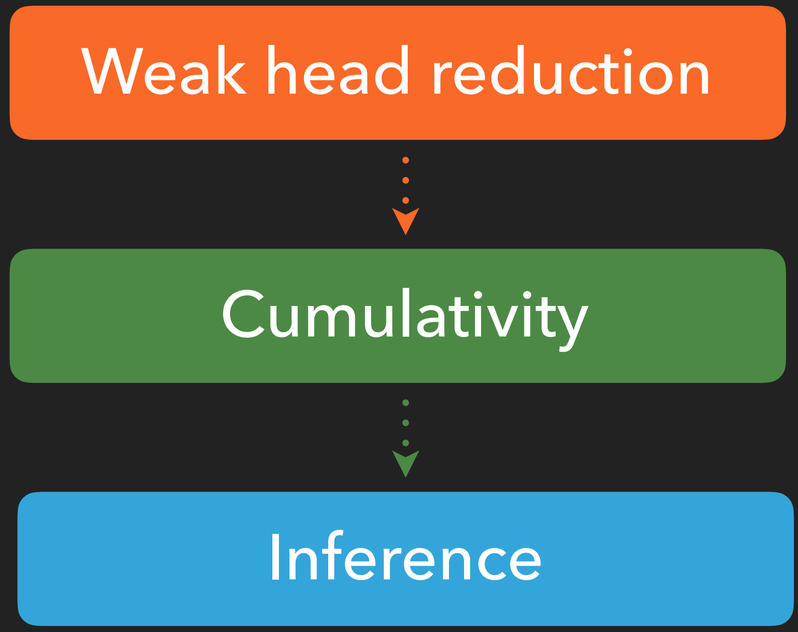


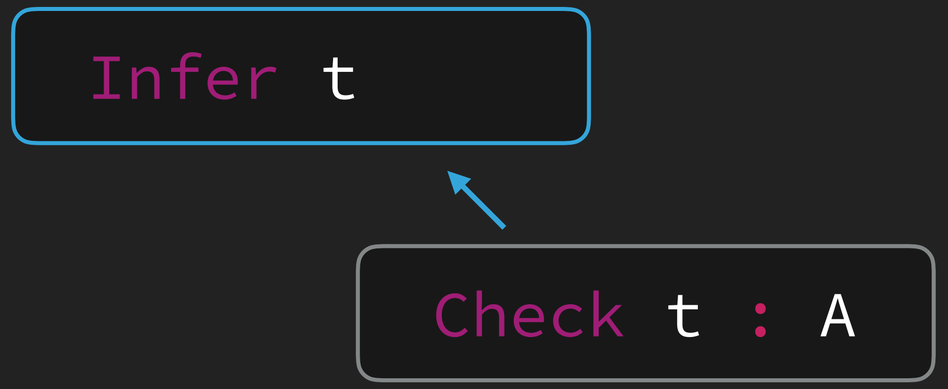
Type Checking



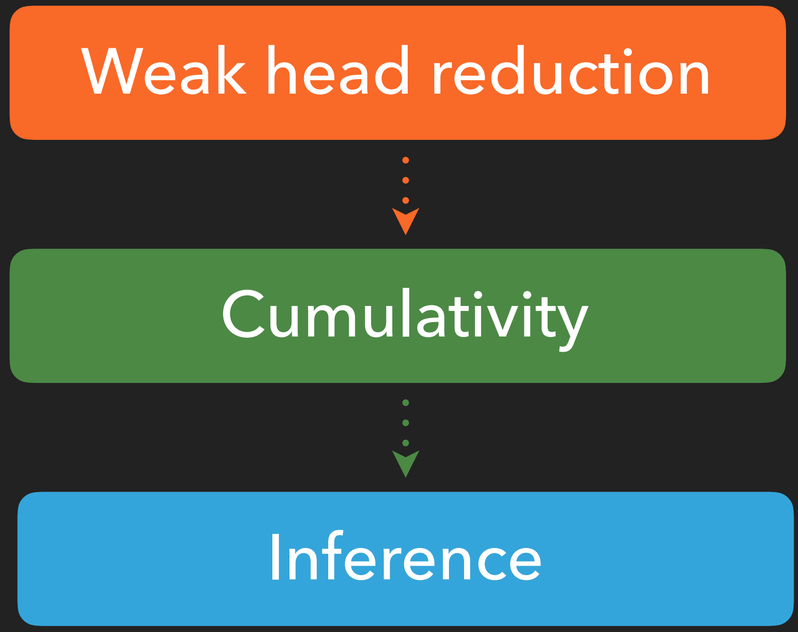

```
Check t : A
```

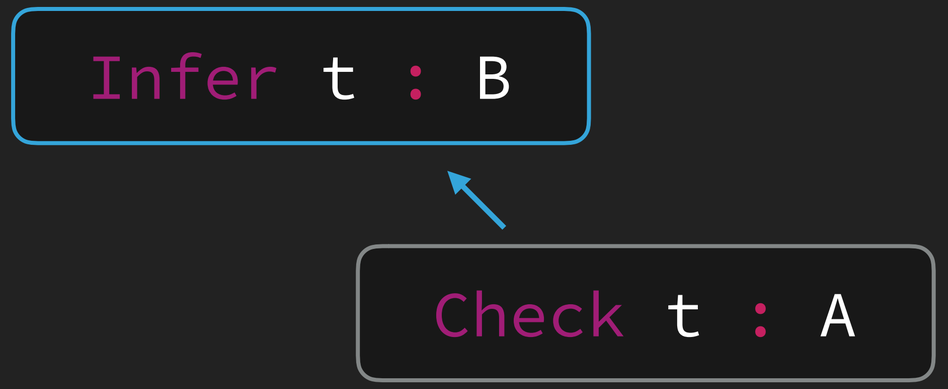
Type Checking



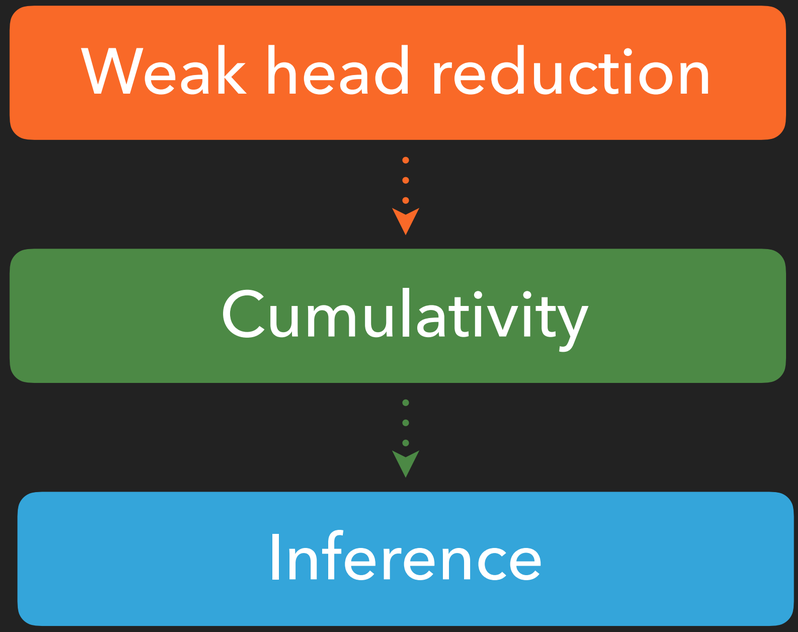


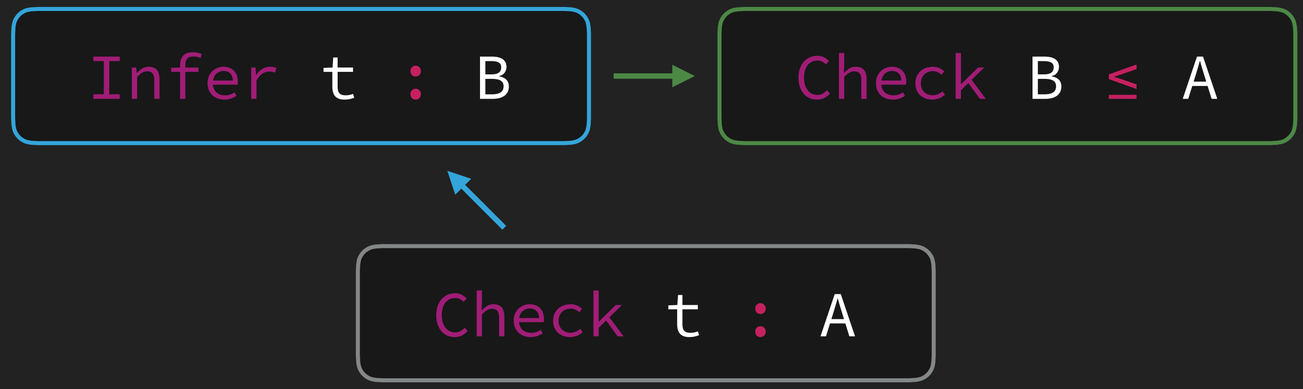
Type Checking



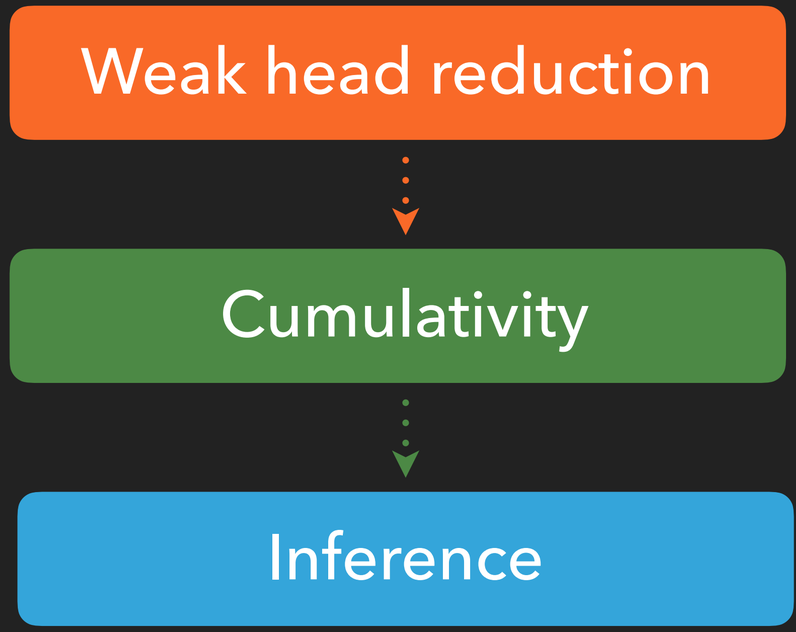


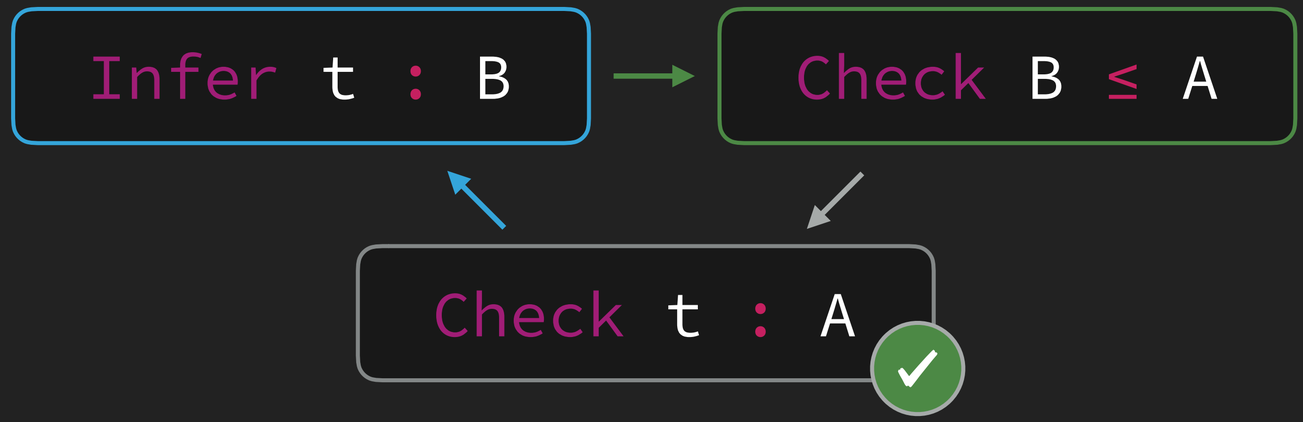
Type Checking





Type Checking





Verifying Erasure

Erasure

At the core of the **extraction** mechanism:

$\mathcal{E} : \text{term} \to \Lambda^{\square,\text{match,fix,cofix}}$

Erases non-computational content:

- Type erasure:

```
Ɛ (t : Type) = □
```

- Proof erasure:

```
Ɛ (p : P : Prop) = □
```

```
fix vrev {A : Type@{i}} {n m : nat} (v : vec A n)
(acc : vec A m) :=
  match v in vec _ n return vec A (n + m) with
  | vnil          ⇒ acc
  | vcons a n v' ⇒
      let idx := S n + m in
      coerce (vec A) idx (e : n + S m = idx)
        (vrev v' (vcons a m acc))
  end.
```

```
Ɛ (vrev) =
```

```
fix vrev n m v acc :=
  match v with
  | vnil          ⇒ acc
  | vcons a n v' ⇒
      let idx := S n + m in
      coerce □ idx □ (vrev v' (vcons a m acc))
  end.
```

Erasure

Singleton elimination principle

Erase propositional content used in computational content:

```
Ɛ (match p in eq _ y with eq_refl ⇒ b end) = Ɛ (b)
```

```
Definition coerce {A} {B : A -> Type) {x} (y : A)
   (e : x = y) : P x -> P y :=
  match e with
  | eq_refl          ⇒ fun p => p
  end.

fix vrev n m v acc :=
  match v with
  | vnil          ⇒ acc
  | vcons a n v' ⇒
      let idx := S n + m in
      coerce □ idx □ (vrev v' (vcons a m acc))
  end.
```

Erasure

Singleton elimination principle

Erase propositional content used in computational content:

```
Ɛ (match p in eq _ y with eq_refl ⇒ b end) = Ɛ (b)
```

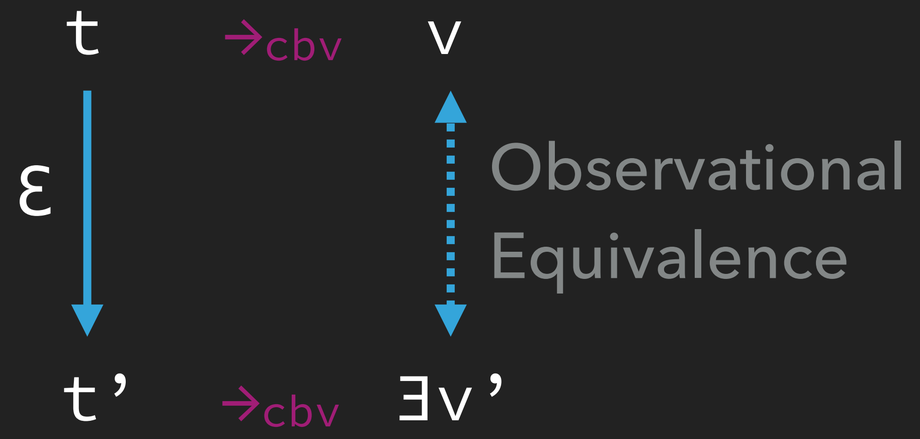
Ɛ (coerce) ~

```
coerce x y := (fun p ⇒ p)
```

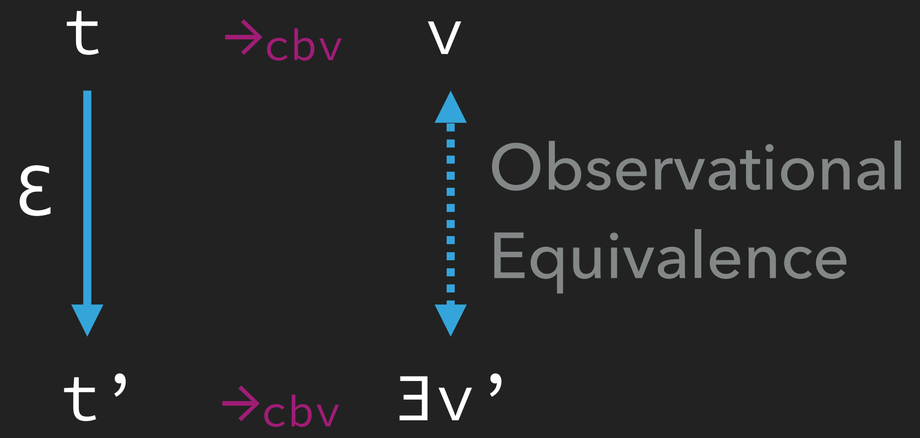
Ɛ (vrev) ~

```
fix vrev n m v acc :=
  match v with
  | vnil          ⇒ acc
  | vcons a n v' ⇒ vrev v' (vcons a m acc)
  end.
```

Erasure Correctness



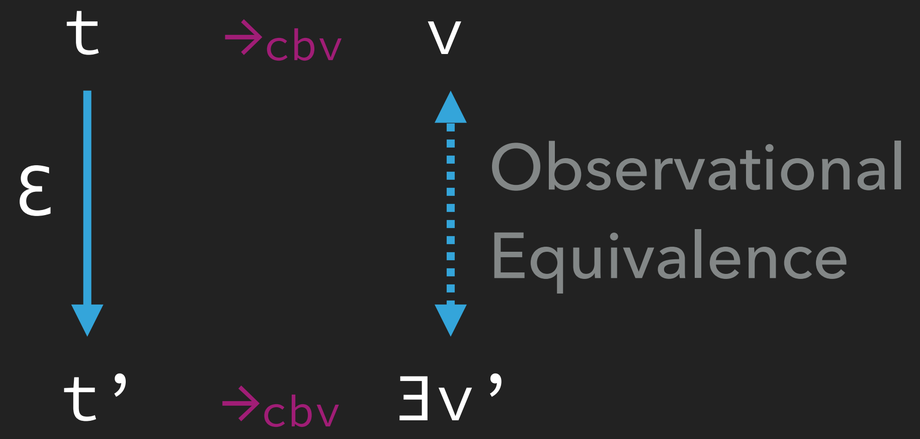
Erasure Correctness



With Canonicity and SN:

$$\vdash t : \mathsf{nat}$$

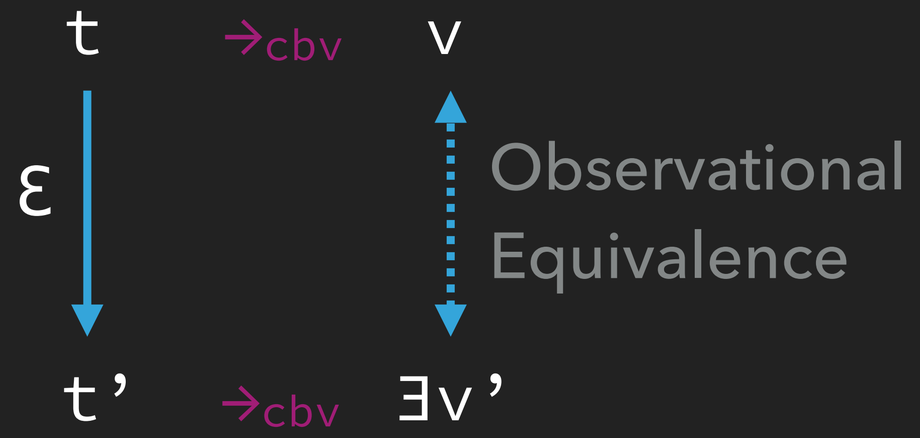
Erasure Correctness



With Canonicity and SN:

```
    ⊢ t : nat
=> ⊢ t → n : nat    (n ∈ ℕ)
```

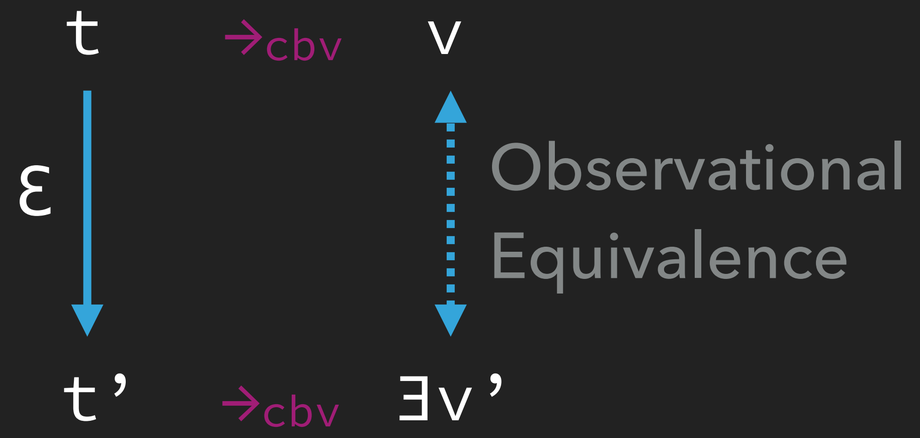
Erasure Correctness



With Canonicity and SN:

```
   ⊢ t : nat
=> ⊢ t → n : nat    (n ∈ ℕ)
=> t →cbv n : nat
```

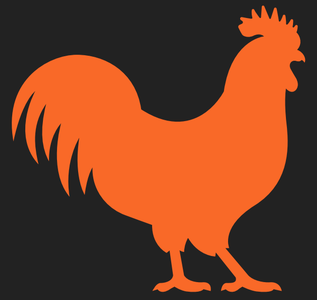
Erasure Correctness



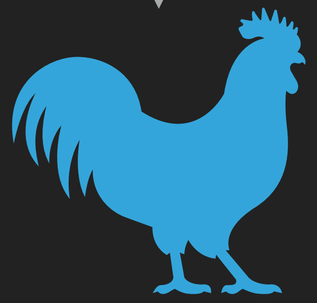
With Canonicity and SN:

```
    ⊢ t : nat
=> ⊢ t → n : nat    (n ∈ ℕ)
=> t →cbv n : nat
=> ε (t) →cbv n
```

Conclusion



Ideal Coq

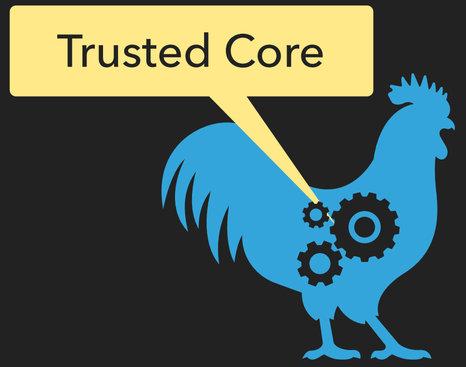


Verified Coq

in

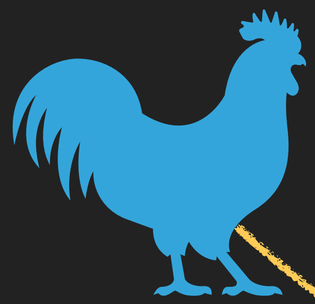
MetaCoq

in



Implemented Coq

Conclusion



Verified Coq

```
MetaCoq Check vrev.
```

Spec: 23kLoC

Proofs: 40kLoC

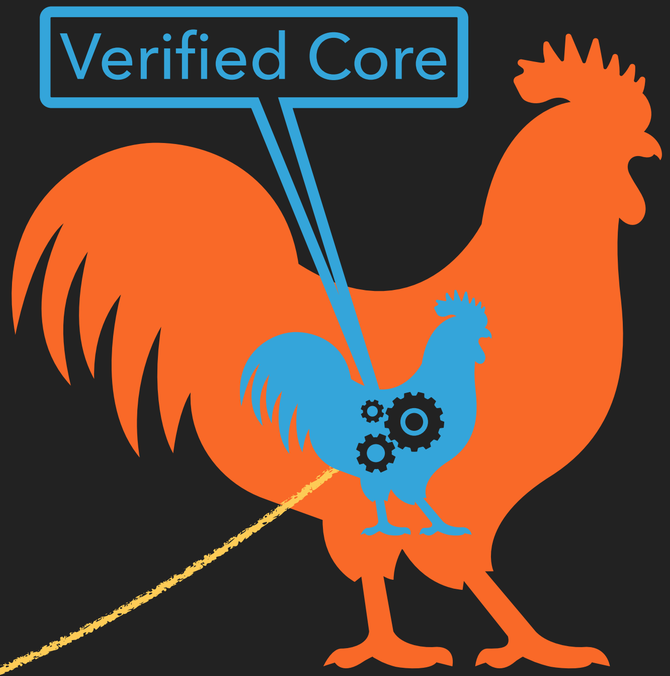
in

MetaCoq

Verified **ε**

```
MetaCoq Erase vrev.
```

in

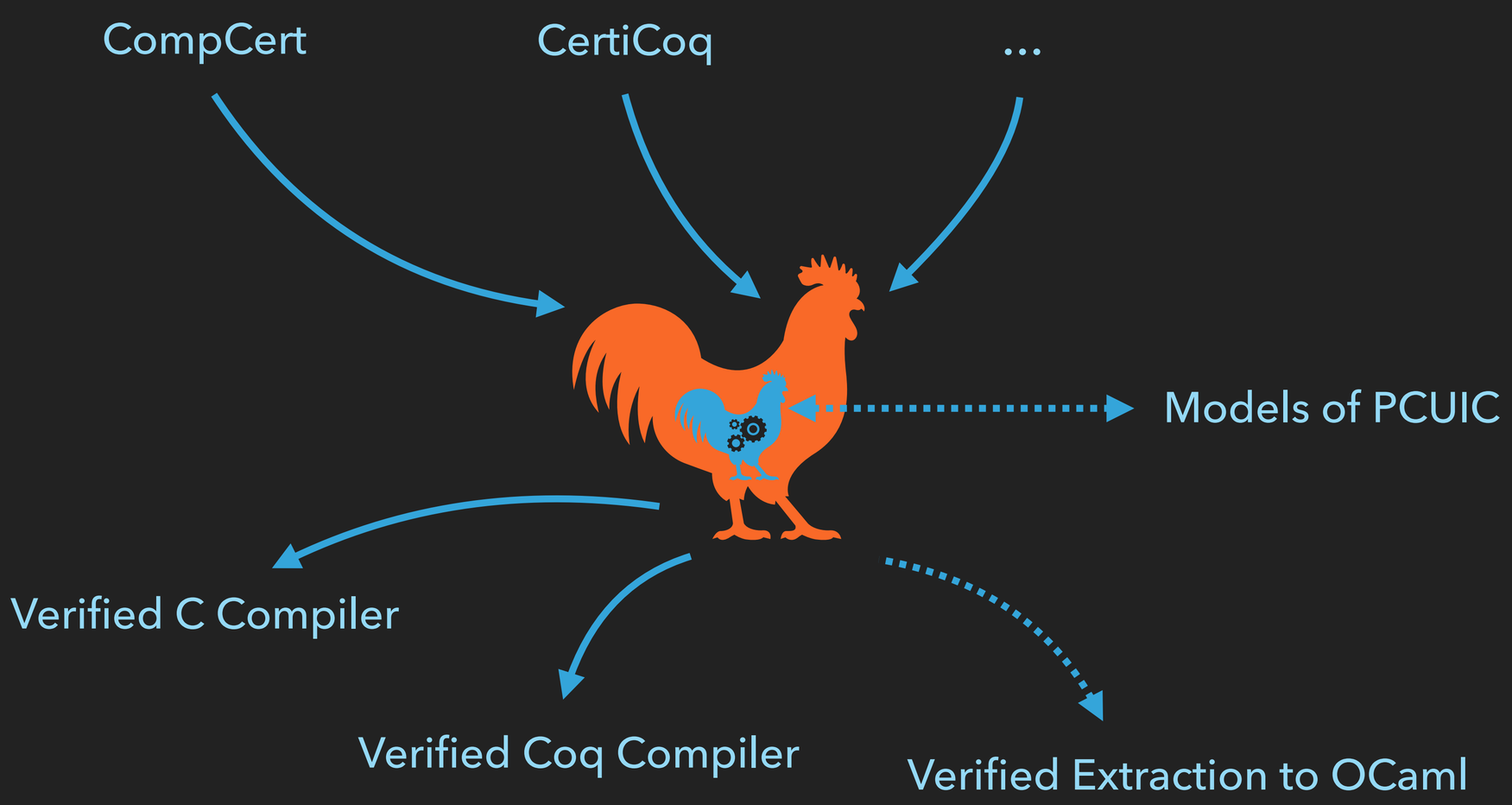


Implemented Coq

=

Ideal Coq

Perspectives



Conclusion

Spec: 23kLoC

Proofs: 40kLoC

https://metacoq.github.io/metacoq

Coq in MetaCoq

« *Cot Cot Codet* ». French, Interjection.

1. Cackle (the cry of a hen, especially one that has laid an egg).

Related Work

- Kumar et al., HOL + CakeML (JAR'16)
- Strub et al., Self-Certification of F* with Coq (POPL'12)