HOR 2025

Controlling computation in type theory

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joint work with

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Main foundations of interactive theorem provers

Higher order logic

Isabelle/HOL

HOL

Main foundations of interactive theorem provers

Higher order logic

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HOL

Dependent type theory

Agda

Lean

Rocq (formerly Coq)

Main foundations of interactive theorem provers

Higher order logic

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HOL

💎 Rewriting 💎

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Dependent type theory

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Computation in type theory

```
refl : 5 + 3 = 10 - 2
```

both sides compute to 8

Computation in type theory

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Types are considered up to computation

```
matrix (0 + x) (2 * y) ≡ matrix x (y + y)
```

Computation in type theory

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definitional equality

Computation in type theory

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propositional equality
(a data type)

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_|_ : matrix n m → matrix n p → matrix n (m + p)
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Assuming `A : matrix 2 4` `B : matrix 2 3`

```
A | B : matrix 2 7
```

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definitional equality

```
_|_ : matrix n m → matrix n p → matrix n (m + p)
```

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

Assuming `A : matrix 2 4` `B : matrix 2 3`

```
A | B : matrix 2 7
```

$$A \mid B = \begin{bmatrix} 0 & 1 & 0 & 2 & 5 & 1 & 4 \\ 1 & 0 & 1 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Proofs by computation ✨

Example: equalities in a monoid

```
ε : M
_·_ : M → M → M
neu_l : ∀ x. ε · x = x
neu_r : ∀ x. x · ε = x
assoc : ∀ x y z. x · (y · z) = (x · y) · z
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How to prove the following?

```
(x · (y · (z · ε))) · (ε · z) = (ε · (x · y)) · (z · z)
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Proofs by computation ✨

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```
rewrite neu_l
```

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`rewrite neu_l, neu_r`

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((x · y) · z) · z = ((x · y) · z) · z
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`rewrite neu_l, neu_r`	`(x · (y · z)) · z = (x · y) · (z · z)`
`rewrite !assoc`	`((x · y) · z) · z = ((x · y) · z) · z`
`reflexivity`	🎉

Proofs by computation ✨

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`reflexivity`	🎉 Yields a proof combining neu_l, neu_r and assoc with congruence of equality

Proofs by computation ✨

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Tedious
but feels like we can easily
compute a normal form

💡 as a list of atoms

Proofs by computation ✨

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```

```
neu : Mexpr
_*_ : Mexpr → Mexpr → Mexpr
`_ : M → Mexpr
```

Syntax of monoid expressions

Proofs by computation ✨

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`norm`

normalisation procedure

```
neu : Mexpr
_*_ : Mexpr → Mexpr → Mexpr
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Syntax of monoid expressions

Proofs by computation ✨

Example: equalities in a monoid

```
eval : Mexpr → M
eval neu := ε
eval (u ⋆ v) := eval u · eval v
eval (` u) := u
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lemma norm_sound : ∀ u v. norm u = norm v → eval u = eval v
```

Proofs by computation ✨

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```

```
(x · (y · (z · ε))) · (ε · z) = (ε · (x · y)) · (z · z)
```

computation

```
eval (`x ⋆ (`y ⋆ (`z ⋆ neu))) ⋆ (neu ⋆ `z) = eval (neu ⋆ (`x ⋆ `y)) ⋆ (`z ⋆ `z))
```

Proofs by computation ✨

Example: equalities in a monoid

norm

normalisation procedure

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eval neu := ε
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neu : Mexpr
_⋆_ : Mexpr → Mexpr → Mexpr
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Syntax of monoid expressions

eval

```
lemma norm_sound : ∀ u v. norm u = norm v → eval u = eval v
```

```
(x · (y · (z · ε))) · (ε · z) = (ε · (x · y)) · (z · z)
```

computation

```
eval (`x ⋆ (`y ⋆ (`z ⋆ neu))) ⋆ (neu ⋆ `z) = eval (neu ⋆ (`x ⋆ `y)) ⋆ (`z ⋆ `z))
```

apply norm_sound

```
norm (`x ⋆ (`y ⋆ (`z ⋆ neu))) ⋆ (neu ⋆ `z) = norm (neu ⋆ (`x ⋆ `y)) ⋆ (`z ⋆ `z))
```

Proofs by computation ✨

Example: equalities in a monoid

norm

normalisation procedure

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eval : Mexpr → M
eval neu := ε
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eval (` u) := u
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neu : Mexpr
_⋆_ : Mexpr → Mexpr → Mexpr
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```

Syntax of monoid expressions

eval

```
lemma norm_sound : ∀ u v. norm u = norm v → eval u = eval v
```

```
(x · (y · (z · ε))) · (ε · z) = (ε · (x · y)) · (z · z)
```

computation

```
eval (`x ⋆ (`y ⋆ (`z ⋆ neu))) ⋆ (neu ⋆ `z) = eval (neu ⋆ (`x ⋆ `y)) ⋆ (`z ⋆ `z))
```

apply norm_sound

```
norm (`x ⋆ (`y ⋆ (`z ⋆ neu))) ⋆ (neu ⋆ `z) = norm (neu ⋆ (`x ⋆ `y)) ⋆ (`z ⋆ `z))
```

computation

```
`x ⋆ (`y ⋆ (`z ⋆ `z)) = `x ⋆ (`y ⋆ (`z ⋆ `z))
```

Proofs by computation ✨

Example: equalities in a monoid (proof comparison)

```
rewrite neu_l
rewrite neu_l, neu_r
rewrite !assoc
reflexivity
```

```
apply norm_sound
reflexivity
```

Proofs by computation ✨

Example: equalities in a monoid (proof comparison)

```
rewrite neu_l
rewrite neu_l, neu_r
rewrite !assoc
reflexivity
```

```
rew (λ X. (x · (y · (z · ε))) · X = (ε · (x · y)) · (z · z)) (neu_l z) (
  rew (λ X. (x · (y · (z · ε))) · z = X · (z · z)) (neu_r (x · y)) (
    rew (λ X. (x · (y · X)) · z = (x · y) · (z · z)) (neu_l z) (
      rew (λ X. X · z = (x · y) · (z · z)) (assoc x y z) (
        rew (λ X. ((x · y) · z) · z = X) (assoc (x · y) z z) refl
      )
    )
  )
)
```

```
apply norm_sound
reflexivity
```

Proofs by computation ✨

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```

doesn't scale 😵‍💫

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apply norm_sound
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    )
  )
)
```

doesn't scale 😵‍💫

```
apply norm_sound
reflexivity
```

```
norm_sound
  (`x * (`y * (`z * neu))) * (neu * `z)
  (neu * (`x * `y)) * (`z * `z))
  refl
```

Proofs by computation ✨

Example: equalities in a monoid (proof comparison)

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      rew (λ X. X · z = (x · y) · (z · z)) (assoc x y z) (
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      )
    )
  )
)
```

doesn't scale 😵‍💫

```
apply norm_sound
reflexivity
```

```
norm_sound
  (`x ⋆ (`y ⋆ (`z ⋆ neu))) ⋆ (neu ⋆ `z)
  (neu ⋆ (`x ⋆ `y)) ⋆ (`z ⋆ `z))
  refl
```

Proof by computation is much shorter and efficient!

(In such a case you can even show completeness of the approach within the ITP itself)

Proofs by computation ✨

Real-life examples

Proofs by computation ✨

Real-life examples

Using reflection to build efficient and certified decision procedures
Boutin
1997

Proving equalities in a commutative ring done right in Coq
Grégoire, Mahboubi
2005

Interpret ring expressions as multivariate polynomials
much more involved and useful than monoids

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Accelerating verified-compiler development with a verified rewrite engine
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2022

Applying the methodology to get a faster
rewrite tactic

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Applying the methodology to get a faster
rewrite tactic

Ongoing experiments with Autosubst
a tool to generate boilerplate about substitution for languages with binders

30′ → 6′
rewrite computation

Mathis Bouverot-Dupuis is working on a new Autosubst-like tool based on computation
(jww Kenji Maillard, Kathrin Stark)

Outline

Controlling computation in type theory

Desirable properties and how to preserve them

Go local!

Limits of computation

(by default)

```
lemma comm : ∀ n m. n + m = m + n
```

Limits of computation

(by default)

```
lemma comm : ∀ n m. n + m = m + n
```

```
induction n
```

Limits of computation

(by default)

```
lemma comm : ∀ n m. n + m = m + n
```

```
induction n
```

```
0 + m = m + 0
```

1

Limits of computation

(by default)

```
lemma comm : ∀ n m. n + m = m + n
```

induction n

```
0 + m = m + 0
```

1

```
ih : ∀ m. n + m = m + n
―――――――――――――――――――――――
S n + m = m + S n
```

2

Limits of computation

(by default)

```
lemma comm : ∀ n m. n + m = m + n
```

```
induction n
```

```
0 + m = m + 0
```

1

```
ih : ∀ m. n + m = m + n
─────────────────────────
S n + m = m + S n
```

2

Limits of computation

(by default)

```
lemma comm : ∀ n m. n + m = m + n
```

induction n

```
0 + m = m + 0
```

1

```
ih : ∀ m. n + m = m + n
-----------------------
S n + m = m + S n
```

2

simpl

```
m = m + 0
```

Limits of computation

(by default)

```
lemma comm : ∀ n m. n + m = m + n
```

induction n

```
0 + m = m + 0
```

1

```
ih : ∀ m. n + m = m + n
S n + m = m + S n
```

2

simpl

```
m = m + 0
```

computation is stuck 😔

Limits of computation

(by default)

```
lemma comm : ∀ n m. n + m = m + n
```

```
induction n    0 + m = m + 0                    1
```

```
ih : ∀ m. n + m = m + n
-----------------------
S n + m = m + S n                               2
```

```
simpl          m = m + 0
```

computation is stuck 😔

```
0 + m ≡ m
S n + m ≡ S (n + m)
```

tension between
definitional and propositional
equality

```
n + 0 = n
n + S m = S (n + m)
```

Limits of computation

(by default)

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lemma comm : ∀ n m. n + m = m + n
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ih : ∀ m. n + m = m + n
───────────────────────
S n + m = m + S n
```

induction n

```
0 + m = m + 0
```

simpl

```
m = m + 0
```

computation is stuck 😔

```
0 + m ≡ m
S n + m ≡ S (n + m)
```

tension between definitional and propositional equality

```
n + 0 = n
n + S m = S (n + m)
```

in vanilla Agda LEAN ROCQ we are forced to choose

Limits of computation

(by default)

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lemma comm : ∀ n m. n + m = m + n
```

induction n

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with custom computation, one can get the best of both worlds

Controlling and extending computation in ITPs

(a very partial history, focusing on implementation)

Definitions by rewriting in the calculus of constructions.

Blanqui

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The Rewster: type-preserving rewrite rules for the Coq proof assistant
Leray, Gilbert, Tabareau, Winterhalter
2024

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better than going through sorted lists

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no need for the usual error handling with monads (or `option`)

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```
lemma hehe : 0 = 1 :=
  raise
```

☣ the logic is now inconsistent ☣

Equality reflection

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$$\frac{\Gamma \vdash p : u =_A v}{\Gamma \vdash u \equiv v}$$

Maximal extensibility

Equality reflection

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need to rely on heuristics *eg* SMT solvers in F*

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Conservative over ITT + UIP + funext

Extensional concepts in intensional type theory
Hofmann
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Effective translation to ITT

Extensionality in the calculus of constructions

Oury

2005

Eliminating reflection from type theory

Winterhalter, Sozeau, Tabareau

2019

Terribly inefficient!

Key properties of dependent type theory

(satisfied by Agda and ROCQ)

Consistency

There is no closed proof of $\bot$, ie there is no term `t` such that

$$\vdash \mathtt{t} : \bot$$

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computation relation

close in spirit to

"Well-typed programs cannot *go wrong*"

– Robin Milner

Deciding conversion

(or definitional equality)

First approximation of conversion:

reflexive, symmetric, transitive closure of computation

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$$\begin{tikzcd}
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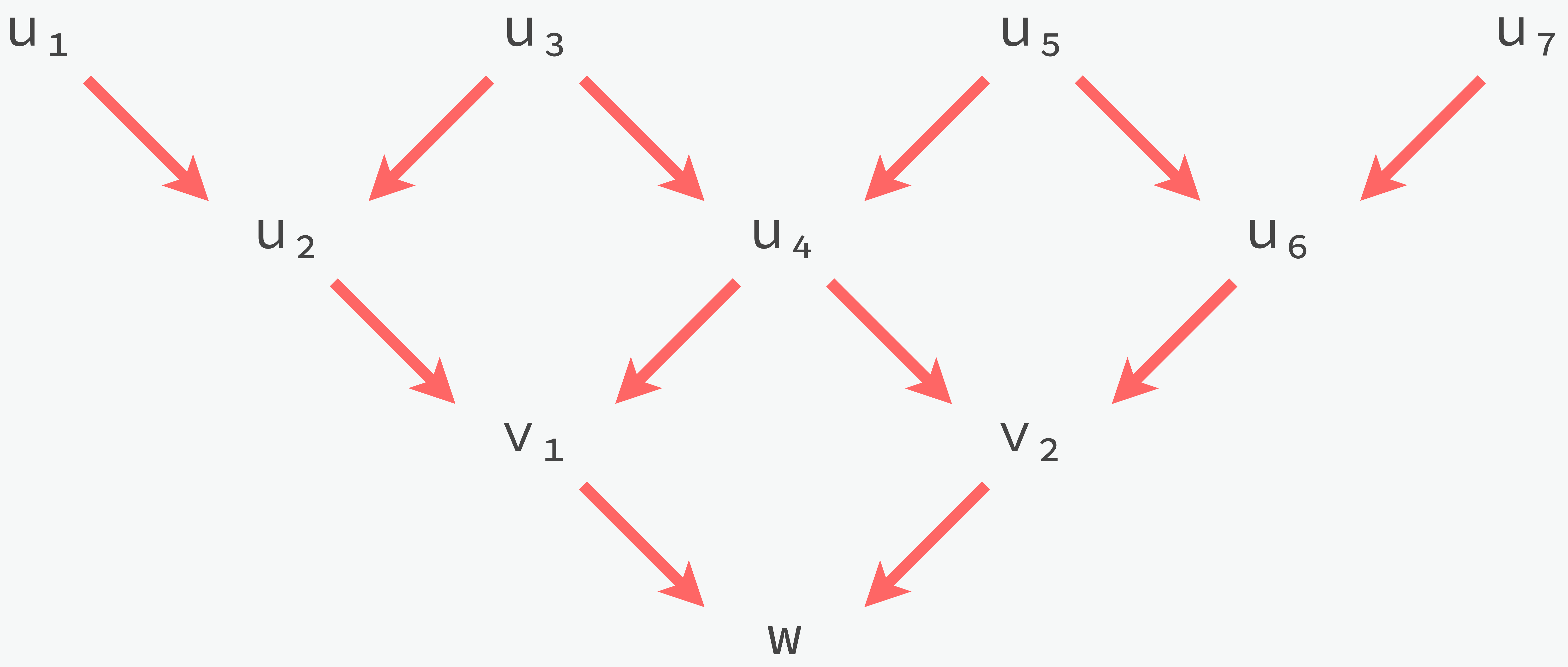
\mathtt{(\lambda x.\ 0\ +\ x)\ (S\ 0)} \arrow[ur] \arrow[r] & \mathtt{0\ +\ S\ 0} \arrow[r] & \mathtt{S\ 0} &
\end{tikzcd}$$

Two properties we *usually* want: confluence and termination

Confluence

If $u \twoheadrightarrow^* v$ and $u \twoheadrightarrow^* w$ then there exists z, such that $v \twoheadrightarrow^* z$ and $w \twoheadrightarrow^* z$



Only computation is necessary to decide conversion!

Termination

(two notions)

Weak normalisation

If $\Gamma \vdash t : A$ then there exist a finite reduction sequence starting from t and ending in an irreducible term.

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In other words there is not infinite reduction sequence starting from `t`.

Hence why it's safe to use whatever reduction strategy: `cbn`, `cbv`, `lazy`…

So we can be smart when choosing how to reduce terms for conversion.
(Indeed, normalisation can be arbitrarily slow.)

How these properties interact with user-defined computation rules

Consistency

$$0 \longrightarrow 1$$

Consistency

$0 \longrightarrow 1$

Same as `Axiom err : 0 = 1`

Consistency

`0` ⟶ `1`

Same as `Axiom err : 0 = 1`

Theorem

If for each rewrite rule `l ⇾ r` we have a proof ⊢ `e : l = r` then the system remains **consistent**.

No need for termination.

Decidability of checking

`loop` $\longrightarrow$ `loop`

Decidability of checking

$$\texttt{loop} \longrightarrow \texttt{loop}$$

breaks termination, hence decidability of checking

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Partial correctness

For **confluent** rewriting systems, the usual algorithm is still correct **if it terminates**!

Type safety

$$\mathbb{N} \to \mathbb{B} \quad \Longrightarrow \quad \mathbb{N} \to \mathbb{N}$$

Type safety

$$\mathbb{N} \to \mathbb{B} \longrightarrow \mathbb{N} \to \mathbb{N}$$

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so $(\lambda x.x)\ 0 \to 0 : \mathbb{N}$

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type safety is lost

$$\begin{tikzcd}
 & A \arrow[dl] \arrow[dr] & \\
\mathbb{N} \to \mathbb{B} & \equiv & \mathbb{N} \to \mathbb{N}
\end{tikzcd}$$

$$\lambda x.x \ : \ \mathbb{N} \to \mathbb{N}$$

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type safety is lost

even without adding rules on arrow types

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ \mathbb{N} \to \mathbb{B} \arrow[dr, dotted] & \equiv & \mathbb{N} \to \mathbb{N} \arrow[dl, dotted] \\ & ? & \end{tikzcd}$$

the key missing property is once again **confluence**

so $(\lambda x.x)\ 0 \to 0 : \mathbb{N}$

type safety is lost

even without adding rules on arrow types

How we establish confluence

Parallel reduction

following Tait, Martin-Löf and Takahashi

$$\rightarrow \;\subset\; \Rrightarrow \;\subset\; \rightarrow^*$$

Can reduce all immediate redexes in one step

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Can reduce all immediate redexes in one step

$$(S\ a) + ((\lambda x.\ x + b)\ 0) \Rrightarrow S\ (a + (0 + b))$$

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$$\texttt{(S a) + ((λx. x + b) 0)} \Rrightarrow \texttt{S (a + (0 + b))}$$

but also

$$\texttt{(S a) + ((λx. x + b) 0)} \Rrightarrow \texttt{(S a) + (0 + b)}$$

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$\rightarrow$ confluent iff $\Rrightarrow$ confluent

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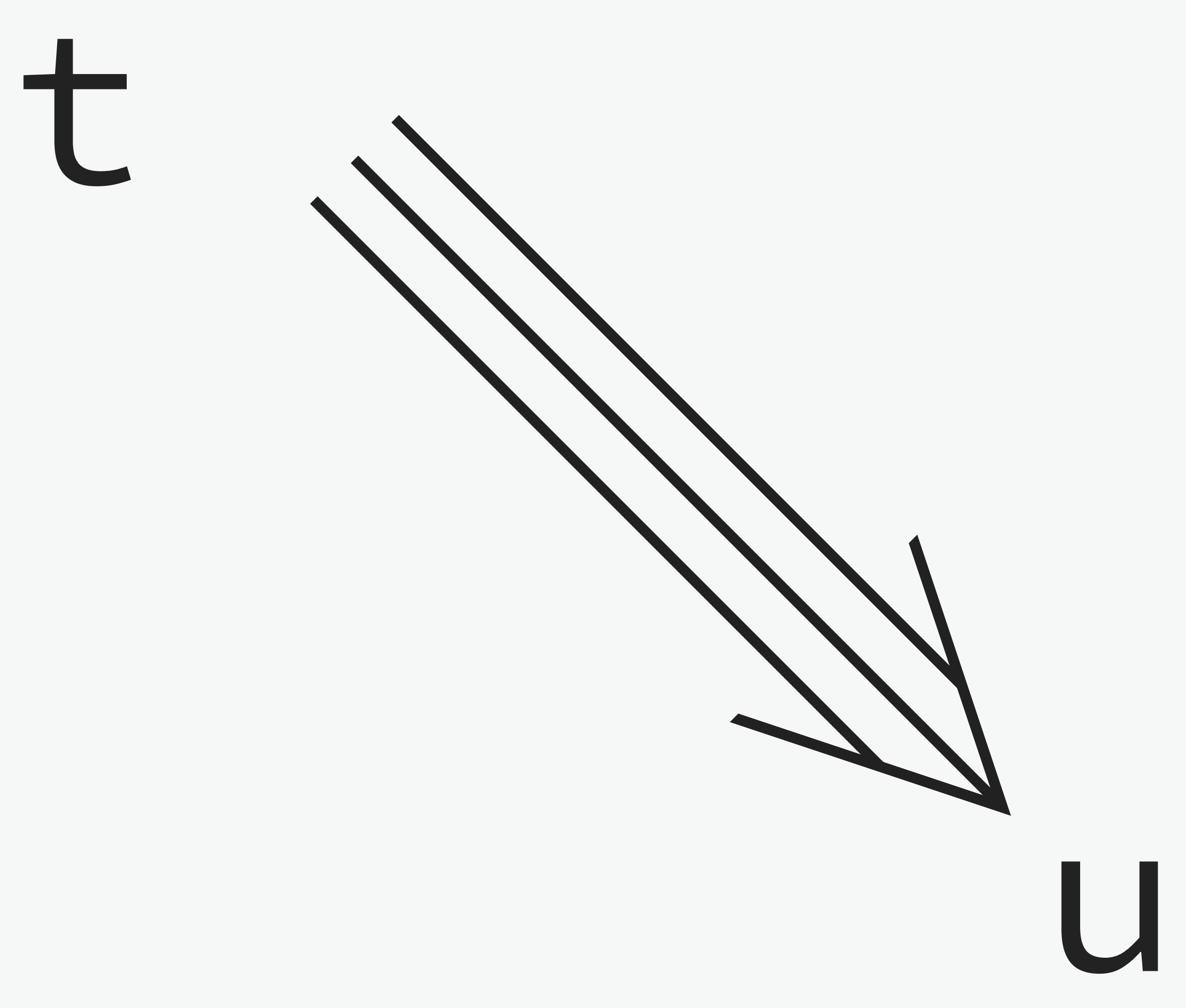
How we establish confluence

Triangle property



How we establish confluence

Triangle property

$$
\begin{tikzcd}
t \arrow[dr, Rrightarrow] \arrow[dd, Rrightarrow, gray] & \\
& u \arrow[dl, Rrightarrow] \\
\rho(t) &
\end{tikzcd}
$$

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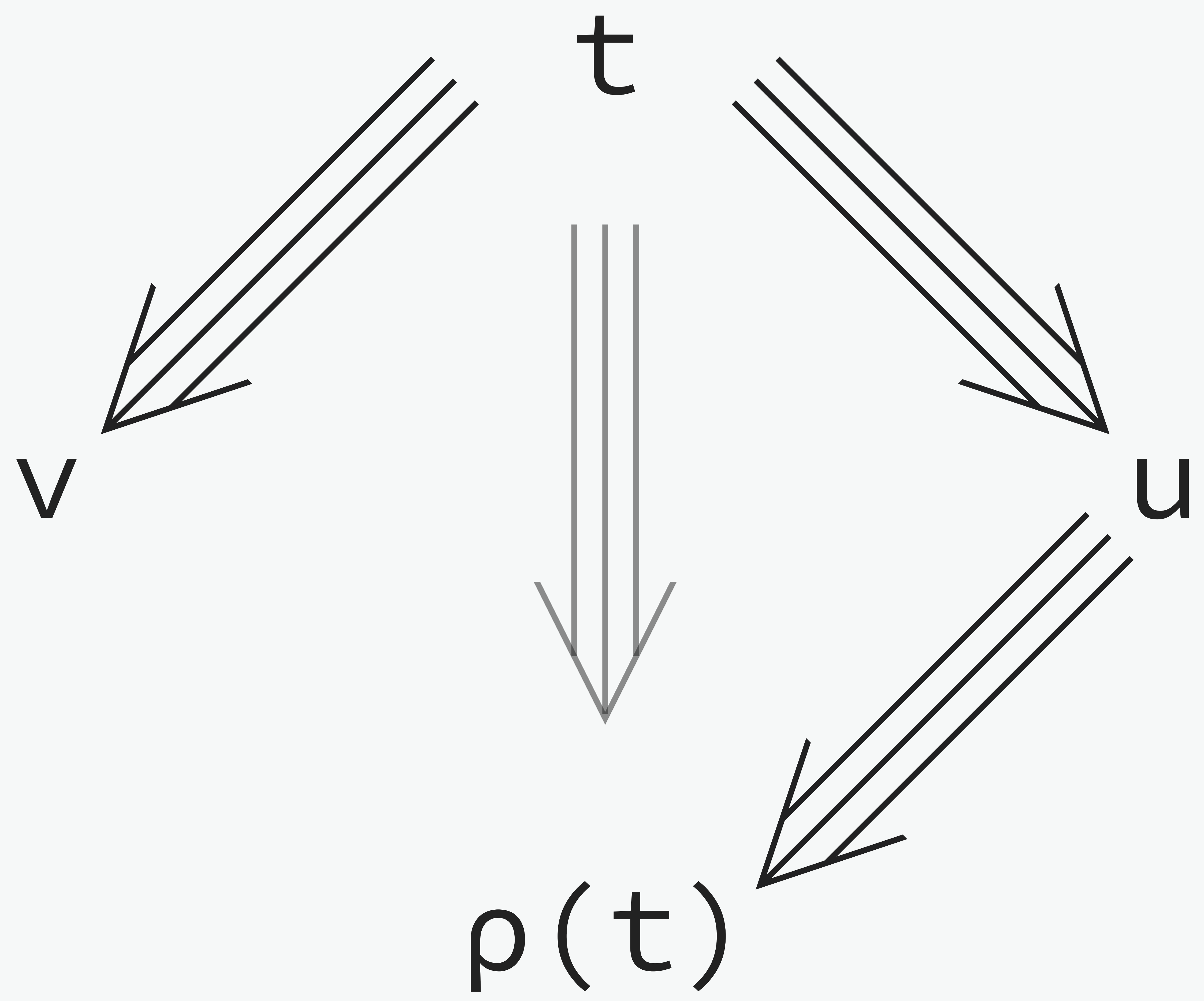

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$$
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How we establish confluence

Triangle property

$$
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\end{tikzcd}
$$

Parallel reduction enjoys the diamond property hence **confluence**

Triangle criterion

the idea

$$\frac{l \rightarrow r \qquad\qquad \sigma \Rrightarrow \sigma'}{l\sigma \Rrightarrow r\sigma'}$$

Triangle criterion

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$$\frac{l \rightarrow r \qquad\qquad \sigma \Rrightarrow \sigma'}{l\sigma \Rrightarrow r\sigma'}$$

```
ρ(lσ) := r(map ρ σ)
```

Triangle criterion

parallel plus example

```
ρ(lσ) := r(map ρ σ)
```

```
0     +    ?y → ?y
?x    +     0 → ?x
S ?x  +    ?y → S (?x + ?y)
?x    + S ?y → S (?x + ?y)
```

Triangle criterion

parallel plus example

$$\rho(\mathrm{S}\ u + \mathrm{S}\ v) := ??$$

```
0     +    ?y → ?y
?x    +     0 → ?x
S ?x  +    ?y → S (?x + ?y)
?x    + S ?y → S (?x + ?y)
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Triangle criterion

parallel plus example

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1. `S ?x + S ?y ⇛ S (S (?x + ?y))`

```
2. 0      +    ?y ⇾ ?y
3. ?x     +     0 ⇾ ?x
4. S ?x +      ?y ⇾ S (?x + ?y)
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Triangle criterion

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Order on the rules (including extra parallel rules) to define ρ

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Every time `lσ` is an instance `l'θ` of an earlier rule, we ask:

$$r\sigma \Rrightarrow r'\theta$$

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Local criterion, checked **modularly**

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Local criterion, checked modularly

Theorem | Confluence holds assuming the above.

Taming user-defined computation

Modular confluence criterion

The Taming of the Rew: A type theory with computational assumptions

Cockx, Tabareau, Winterhalter

2021

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+ many tools developed by the community!

Go local!

Why locality matters

Example: exceptions

```
symb raise : ∀ {A}. A
rule if raise then t else f ↦ raise
def nth_exn : list A → ℕ → A := …
```

Agda Rocq

Computation rules must be assumed forever

Why locality matters

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Agda · Lean · ROCQ

Computation rules must be assumed **forever**

No way to ensure the rules aren't used here

(unlike axioms, there is no `Print Assumptions`)

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Rules extend the trusted computing base

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Rules extend the **trusted computing base**

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Overall, not very **modular**
(or type-theoretic)

Our proposal

Prenex quantification over computation rules

```
interface Bool
  assumes
    bool : Type
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    ifte P t f true ⇾ t
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Equations are verified implicitly

You can exploit the encoding without having to work directly with it!

Example: Strictifying substitutions

and renamings

Autosubst encodes substitutions as functions $\mathbb{N} \to$ `term` and renamings as $\mathbb{N} \to \mathbb{N}$
to get definitional unitality and associativity of composition

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and so on...

Other examples include…

Hiding implementation details
while retaining computation

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interface Shift
  assumes
    shift : list ℕ → list ℕ
  where
    shift (x :: l) ⇒ S x :: shift l
    shift [] ⇒ []
```

```
instance ShiftasMap
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```

This way, `map` never appears in goals out of nowhere
useful for automatically generated functions (eg. Equations in Rocq)

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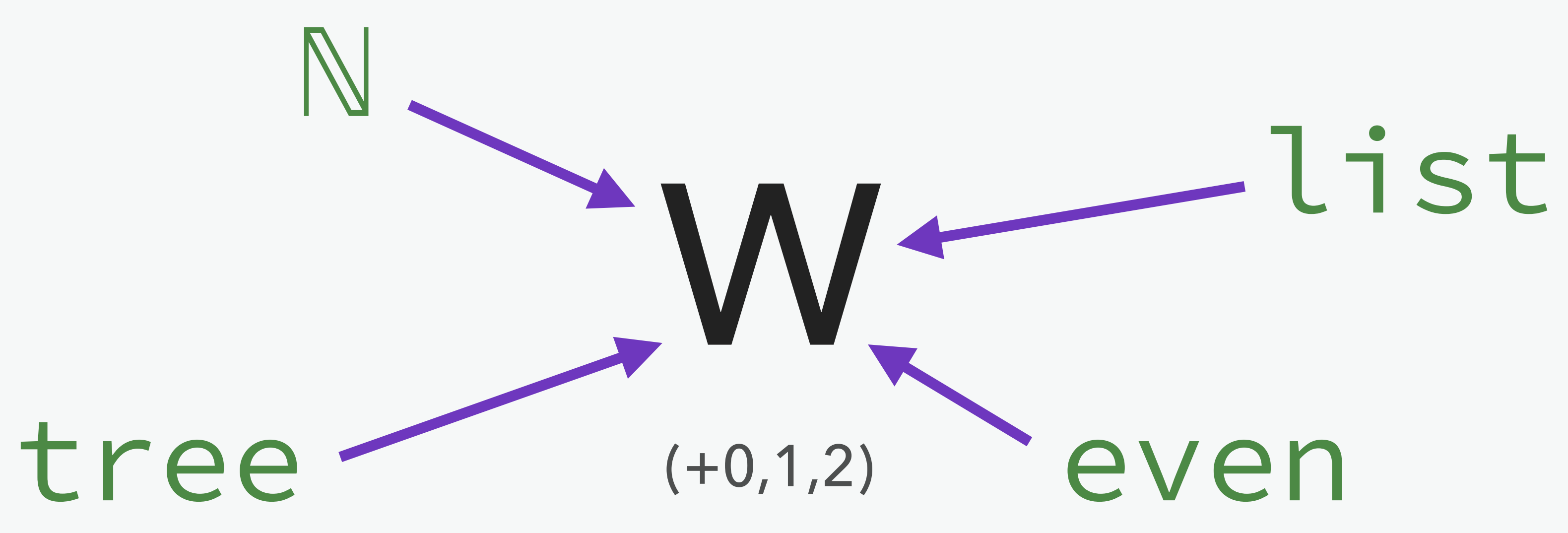

Why not W?

Hugunin

2021

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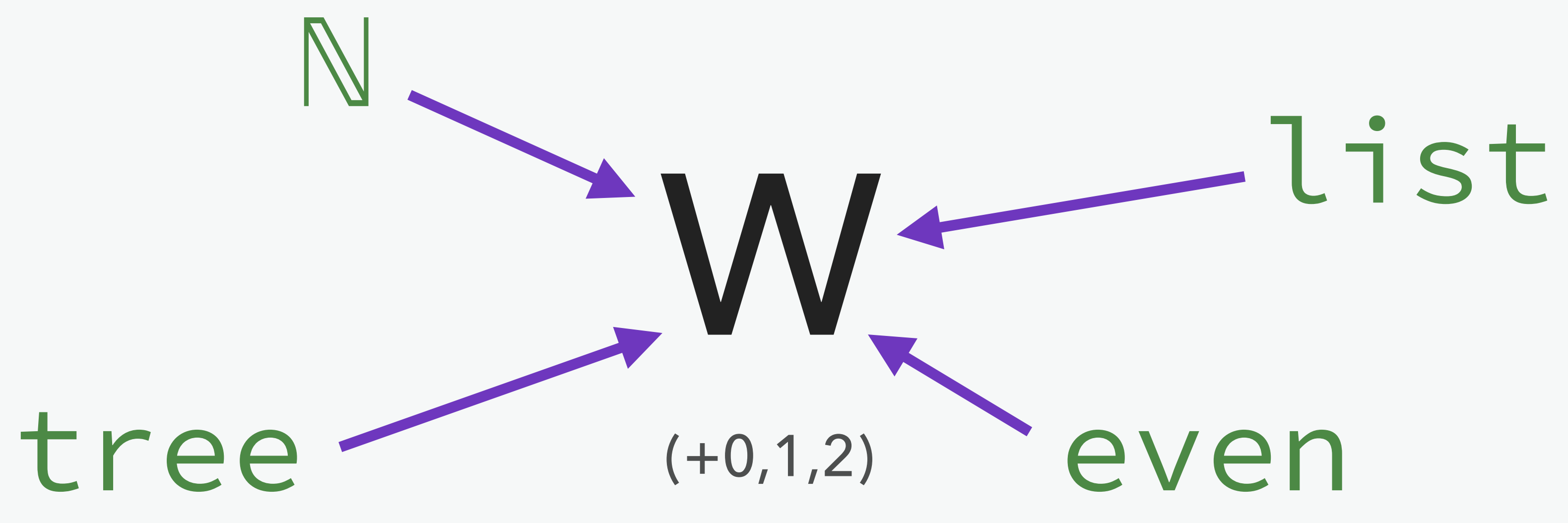

Why not W?

Hugunin

2021

Use effects locally, eg. exceptions

The type theory: LRTT

$$\Sigma \mid \Xi \mid \Gamma \vdash t : A$$

The type theory: LRTT

```
symb x : A
```

```
rule l ↪ r
```

Interface environment

$$\Sigma \mid \Xi \mid \Gamma \vdash t : A$$

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Global environment

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Basically regular MLTT

$$\Sigma \mid \Xi \mid \boxed{\Gamma \vdash t : A}$$

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Basically regular MLTT

$$\Sigma \mid \Xi \mid \boxed{\Gamma \vdash t : A}$$

Computation rule

$$\frac{(l \rightarrow r) \in \Xi}{\Sigma \mid \Xi \mid \Gamma \vdash l[\sigma] \equiv r[\sigma]}$$

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Computation rule

$$\frac{(l \to r) \in \Xi}{\Sigma \mid \Xi \mid \Gamma \vdash l[\sigma] \equiv r[\sigma]}$$

Unfolding rule

$$\frac{(\mathtt{def}\ f\langle \Xi' \rangle : A := t) \in \Sigma \qquad \Sigma \mid \Xi \mid \Gamma \vdash \xi : \Xi'}{\Sigma \mid \Xi \mid \Gamma \vdash f\langle \xi \rangle \equiv t\xi}$$

Meta-theory

mostly usual

Environment weakening (Σ, Ξ, Γ), substitution, instantiation, validity

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```
Σ | Ξ | Γ ⊢ ξ : Ξ' →
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Meta-theory

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Consistency

A given by embedding into a theory
with equality reflection

Meta-theory

mostly usual

Environment weakening (Σ, Ξ, Γ), **substitution, instantiation, validity**

$\Sigma \mid \Xi \mid \Gamma \vdash \xi : \Xi' \to$
$\Sigma \mid \Xi' \mid \cdot \vdash t : A \to$
$\Sigma \mid \Xi \mid \Gamma \vdash t\xi : A\xi$

Consistency

A given by embedding into a theory
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more interesting:

Conservativity over MLTT

$\cdot \mid \cdot \mid \cdot \vdash A : \mathsf{Type} \to$
$\Sigma \mid \cdot \mid \cdot \vdash t : A \to$
$\exists\ t'.\ \cdot \mid \cdot \mid \cdot \vdash t' : A$

Obtained by **inlining** definitions

Inlining

if $\Sigma \mid \Xi \mid \Gamma \vdash t : A$

then $\cdot \mid [\![\Xi]\!]\langle \kappa \rangle \mid [\![\Gamma]\!]\langle \kappa \rangle \vdash [\![t]\!]\langle \kappa \rangle : [\![A]\!]\langle \kappa \rangle$

where κ interprets the definitions of Σ

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where κ interprets the definitions of Σ

with κ fixed (and abstract):

$[\![x]\!] := x$

$[\![\lambda\,(x : A).\ t]\!] := \lambda\,(x : [\![A]\!]).\ [\![t]\!]$

$[\![u\ v]\!] := [\![u]\!]\ [\![v]\!]$

$[\![M.x]\!] := M.x$

$\boxed{[\![f\langle \xi \rangle]\!] := (\kappa\ f)[\![\xi]\!]}$

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κ is then defined by induction on ⊢ Σ such that

when $(\mathrm{def}\ f\langle\ \Xi'\ \rangle : A := t) \in \Sigma$ we have $\kappa\ f := [\![t]\!]\langle\ \kappa\_rec\ \rangle$

Inlining

if $\Sigma \mid \Xi \mid \Gamma \vdash t : A$

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Inlining

Compared to conservativity, we need full generality here

if $\Sigma \mid \Xi \mid \Gamma \vdash t : A$

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Characterising conversion with reduction

proof-of-concept confluence proof similar to the one presented earlier

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$$\begin{tikzcd}
\Sigma \mid \Xi \vdash u \equiv v \arrow[rr] & & \Sigma \mid \Xi \vdash u \leftrightsquigarrow v \arrow[dl, dashed, "\text{assuming confluence}"] \\
& \Sigma \mid \Xi \vdash u \bowtie v \arrow[ul, dashed, "\text{starting from well-typed terms}"] &
\end{tikzcd}$$

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& \Sigma \mid \Xi \vdash u \bowtie v \arrow[ul, dashed, "\text{starting from well-typed terms}"] &
\end{tikzcd}$$

Injectivity of Π-types

if `Π (x : A). B ≡ Π (x : C). D` then `A ≡ C` and `B ≡ D`

Checking for conversion

Characterising conversion with reduction

proof-of-concept confluence proof similar to the one presented earlier

$$
\begin{tikzcd}
\Sigma \mid \Xi \vdash u \equiv v \arrow[rr] & & \Sigma \mid \Xi \vdash u \leftrightsquigarrow v \arrow[dl, dashed, "\text{assuming confluence}"] \\
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Subject reduction

assuming rewrite rules preserve types and are confluent

ROCQ prototype

on top of

The Rewster: type-preserving rewrite rules for the Coq proof assistant

Leray, Gilbert, Tabareau, Winterhalter

2024

ROCQ prototype

on top of

The Rewster: type-preserving rewrite rules for the Coq proof assistant

Leray, Gilbert, Tabareau, Winterhalter

2024

using the module system for interfaces and instances
and functors to quantify over interfaces

```
Module Type Plus.
  Symbol plus : ℕ → ℕ → ℕ.
  Rewrite Rules plus_r :=
  | plus 0 ?n => ?n
  | plus (S ?m) ?n => S (plus ?m ?n).
End Plus.
```

ROCQ prototype

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Module UsePlus (P : Plus).
  Import P.

  Compute (plus 2 3). (* 5 *)

End UsePlus.
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ROCQ prototype

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Module UsePlus (P : Plus).
  Import P.

  Compute (plus 2 3). (* 5 *)

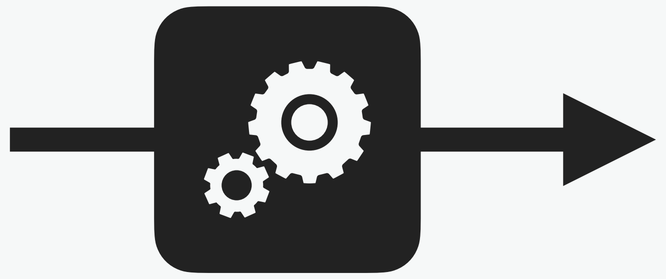
End UsePlus.
```

```
Module Import PlusImpl : Plus.
  Definition plus := Nat.add.
End PlusImpl.
```

Conclusion

Conservative extension of MLTT with **local computation**

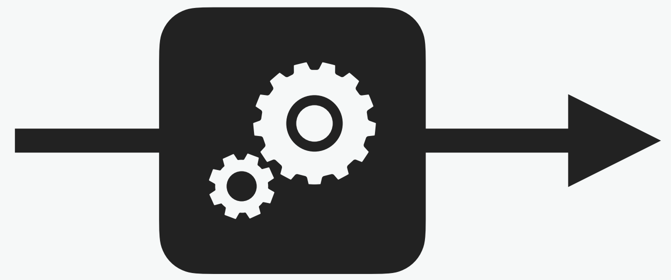
Conclusion



hide implementation details

Conservative extension of MLTT with **local computation**

Conclusion



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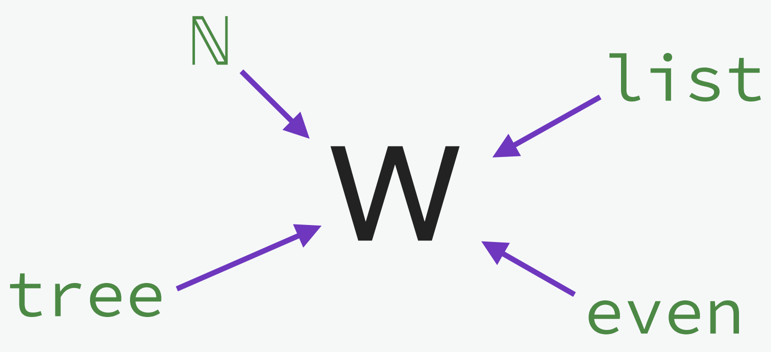
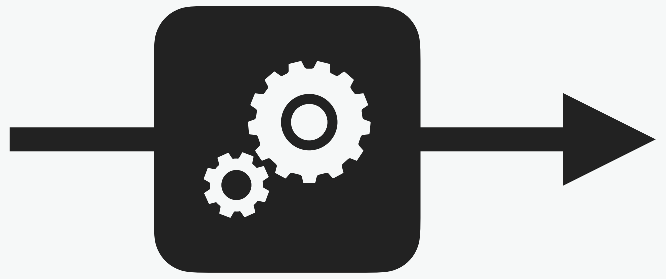


encode features

Conservative extension of MLTT with **local computation**

Conclusion



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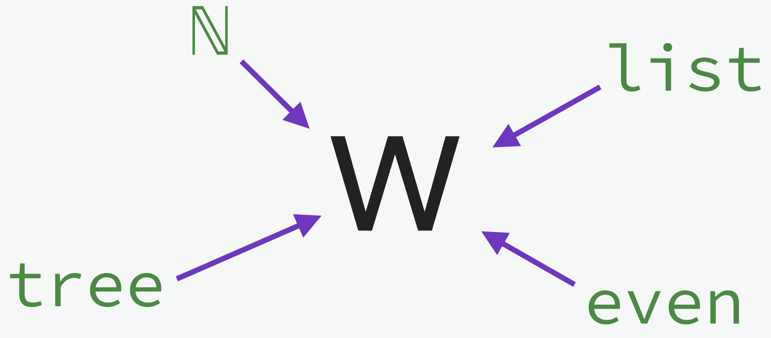
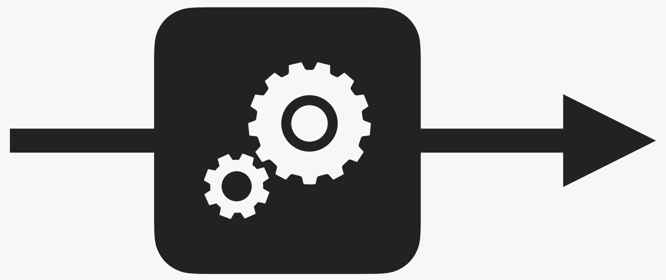


encode features

contained extensions (safer)

Conservative extension of MLTT with **local computation**

Conclusion



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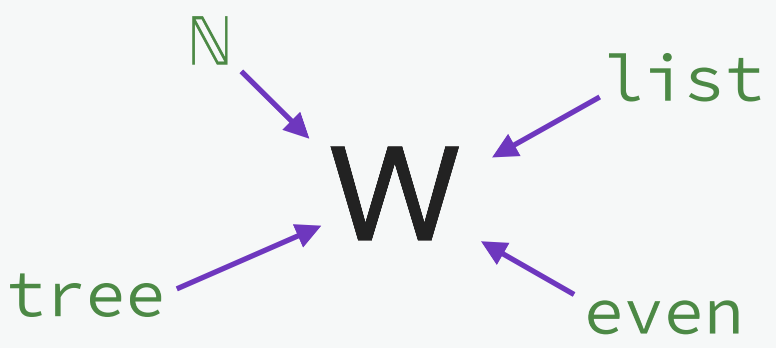

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Conservative extension of MLTT with **local computation**

ROCQ APPROVED

/TheoWinterhalter/local-comp

Conclusion

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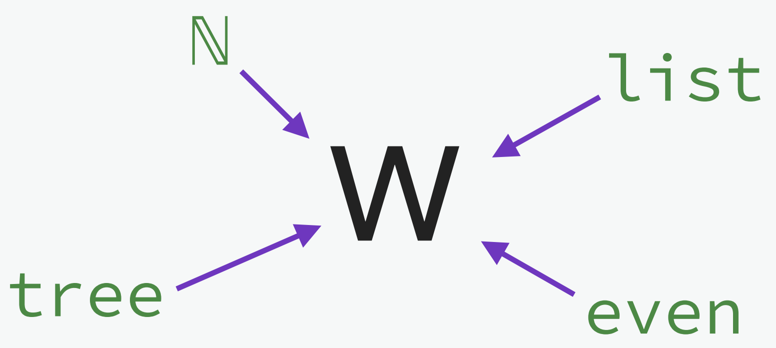

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contained extensions (safer)

Conservative extension of MLTT with **local computation**

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/TheoWinterhalter/local-comp

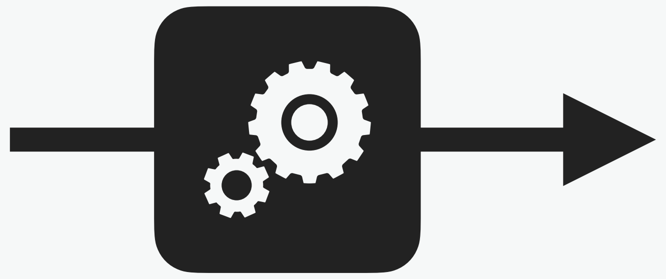
Perspectives (ongoing)

ROCQ

Concrete implementations

Agda

Conclusion



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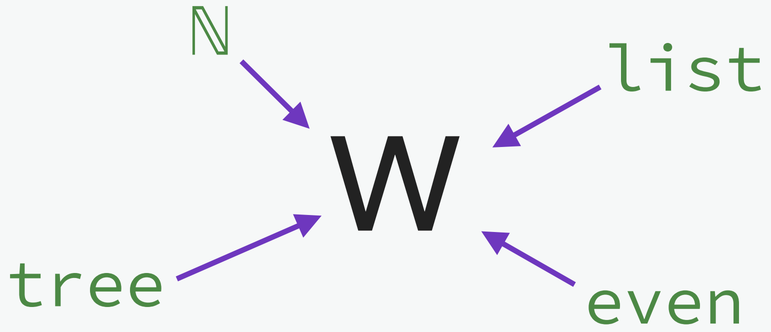

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contained extensions (safer)

Conservative extension of MLTT with **local computation**

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Perspectives (ongoing)

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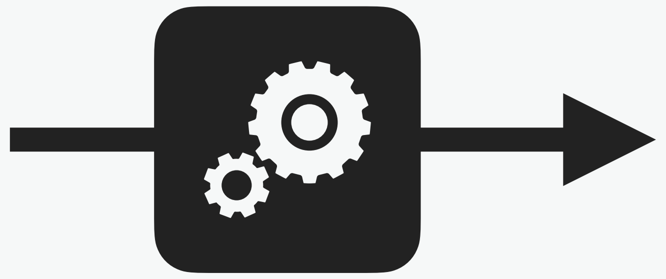
Propositional instances

Conclusion



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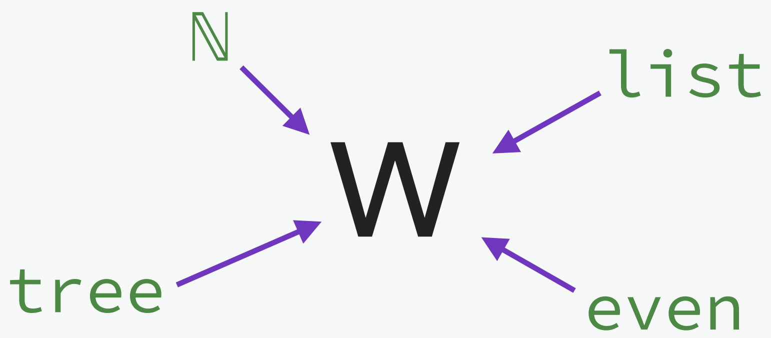

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Conservative extension of MLTT with **local computation**

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Propositional instances

Thank you!