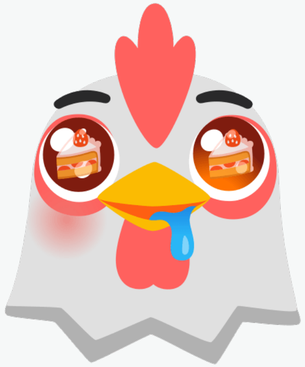
25 March 2026

Proofs and Algorithms seminar

Encode the Cake and Eat it Too

Controlling computation in type theory, locally



Théo Winterhalter

joint work with Yann Leray

Main foundations of interactive theorem provers

Higher order logic

Isabelle/HOL

HOL

Dependent type theory

Agda

Lean

Rocq (formerly Coq)

Main foundations of interactive theorem provers

Higher order logic

Isabelle/HOL

HOL

Focus of my research 🔬 ⟶ **Dependent type theory**

Agda

Lean

Rocq (formerly Coq)

Computation in type theory

```
refl : 5 + 3 = 10 - 2
```

both sides reduce to 8

Computation in type theory

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refl : 5 + 3 = 10 - 2
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Types are considered up to computation

```
matrix (0 + x) (2 * y) ≡ matrix x (y + y)
```

Computation in type theory

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refl : 5 + 3 = 10 - 2
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matrix (0 + x) (2 * y) ≡ matrix x (y + y)
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definitional equality

Computation in type theory

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both sides reduce to 8

propositional equality
(a data type)

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_|_ : matrix n m → matrix n p → matrix n (m + p)
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Assuming `A : matrix 2 4` `B : matrix 2 3`

```
A | B : matrix 2 7
```

Computation in type theory

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Assuming `A : matrix 2 4` `B : matrix 2 3`

```
A | B : matrix 2 7
```

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A \mid B = \begin{bmatrix} 0 & 1 & 0 & 2 & 5 & 1 & 4 \\ 1 & 0 & 1 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Proofs by computation ✨

Example: equalities in a monoid

```
ε : M
_·_ : M → M → M
neu_l : ∀ x. ε · x = x
neu_r : ∀ x. x · ε = x
assoc : ∀ x y z. x · (y · z) = (x · y) · z
```

Proofs by computation ✨

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How to prove the following? `(x · (y · (z · ε))) · (ε · z) = (ε · (x · y)) · (z · z)`

Proofs by computation ✨

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(x · (y · (z · ε))) · (ε · z) = (ε · (x · y)) · (z · z)
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```
rewrite neu_l
```

Proofs by computation ✨

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rewrite neu_l, neu_r
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Proofs by computation ✨

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(x · (y · z)) · z = (x · y) · (z · z)
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((x · y) · z) · z = ((x · y) · z) · z
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Proofs by computation ✨

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`reflexivity` 🎉

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`reflexivity` 🎉 Yields a proof combining neu_l, neu_r and assoc with congruence of equality

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`reflexivity` 🎉 Yields a proof combining neu_l, neu_r and assoc with congruence of equality

Tedious
but feels like we can easily
compute a normal form

💡 as a list of atoms

Proofs by computation ✨

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```
neu : Mexpr
_*_ : Mexpr → Mexpr → Mexpr
`_ : M → Mexpr
```

Syntax of monoid expressions

Proofs by computation ✨

Example: equalities in a monoid

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```

`norm`

normalisation procedure

```
neu : Mexpr
_*_ : Mexpr → Mexpr → Mexpr
`_ : M → Mexpr
```

Syntax of monoid expressions

Proofs by computation ✨

Example: equalities in a monoid

```
eval : Mexpr → M
eval neu := ε
eval (u * v) := eval u · eval v
eval (` u) := u
```

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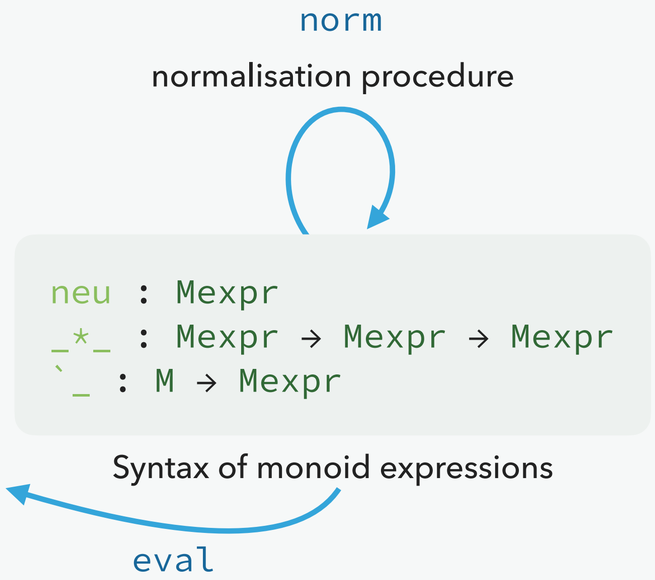

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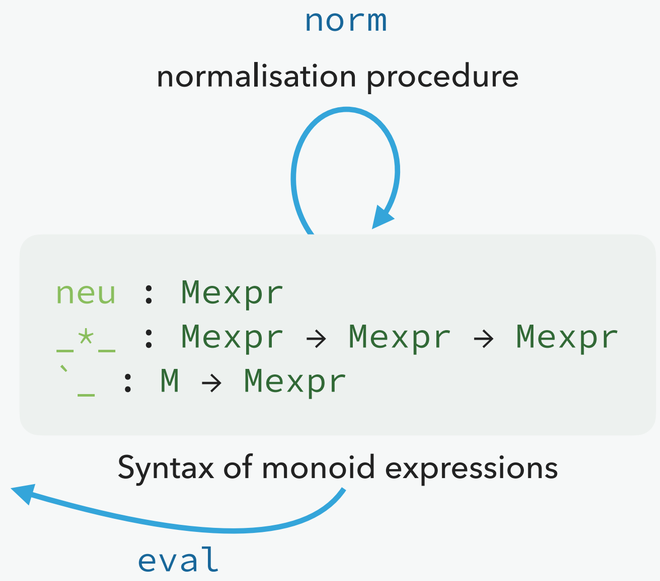


```
lemma norm_sound : ∀ u v. norm u = norm v → eval u = eval v
```

Proofs by computation ✨

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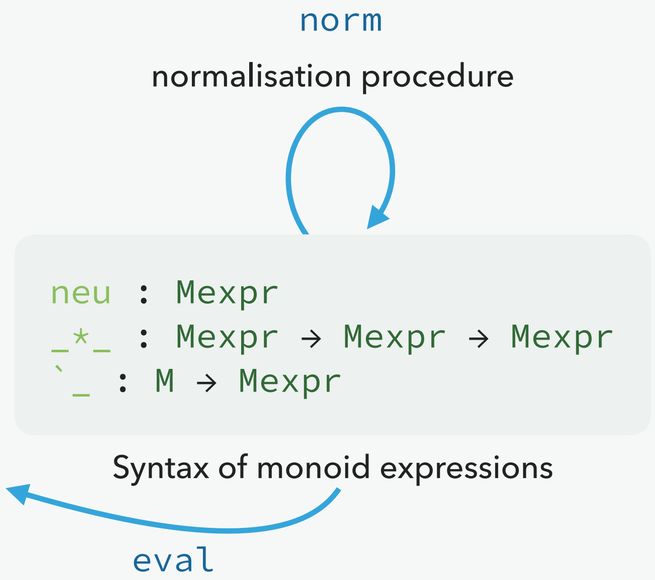


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Syntax of monoid expressions

eval

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lemma norm_sound : ∀ u v. norm u = norm v → eval u = eval v
```

computation

```
(x · (y · (z · ε))) · (ε · z) = (ε · (x · y)) · (z · z)
```

```
eval (`x * (`y * (`z * neu))) * (neu * `z) = eval (neu * (`x * `y)) * (`z * `z))
```

Proofs by computation ✨

Example: equalities in a monoid

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Syntax of monoid expressions

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normalisation procedure

eval

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```

computation

```
(x · (y · (z · ε))) · (ε · z) = (ε · (x · y)) · (z · z)
```

```
eval (`x * (`y * (`z * neu))) * (neu * `z) = eval (neu * (`x * `y)) * (`z * `z))
```

```
norm (`x * (`y * (`z * neu))) * (neu * `z) = norm (neu * (`x * `y)) * (`z * `z))
```

apply norm_sound

Proofs by computation ✨

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```
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computation

```
eval (`x * (`y * (`z * neu))) * (neu * `z) = eval (neu * (`x * `y)) * (`z * `z))
```

apply norm_sound

```
norm (`x * (`y * (`z * neu))) * (neu * `z) = norm (neu * (`x * `y)) * (`z * `z))
```

computation

```
`x * (`y * (`z * `z)) = `x * (`y * (`z * `z))
```

Proofs by computation ✨

Example: equalities in a monoid (proof comparison)

```
rewrite neu_l
rewrite neu_l, neu_r
rewrite !assoc
reflexivity
```

```
apply norm_sound
reflexivity
```

Proofs by computation ✨

Example: equalities in a monoid (proof comparison)

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rewrite neu_l
rewrite neu_l, neu_r
rewrite !assoc
reflexivity
```

```
rew (λ X. (x · (y · (z · ε))) · X = (ε · (x · y)) · (z · z)) (neu_l z) (
  rew (λ X. (x · (y · (z · ε))) · z = X · (z · z)) (neu_r (x · y)) (
    rew (λ X. (x · (y · X)) · z = (x · y) · (z · z)) (neu_l z) (
      rew (λ X. X · z = (x · y) · (z · z)) (assoc x y z) (
        rew (λ X. ((x · y) · z) · z = X) (assoc (x · y) z z) refl
      )
    )
  )
)
```

```
apply norm_sound
reflexivity
```

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doesn't scale 😵‍💫

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```
apply norm_sound
reflexivity
```

```
norm_sound
  (`x * (`y * (`z * neu))) * (neu * `z)
  (neu * (`x * `y)) * (`z * `z))
  refl
```

Proofs by computation ✨

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apply norm_sound
reflexivity
```

```
norm_sound
  (`x * (`y * (`z * neu))) * (neu * `z)
  (neu * (`x * `y)) * (`z * `z))
  refl
```

Proof by computation is much shorter and efficient!

(In such a case you can even show completeness of the approach within the ITP itself)

Proofs by computation ✨

Real-life examples

Proofs by computation ✨

Real-life examples

Using reflection to build efficient and certified decision procedures

Boutin

1997

Proving equalities in a commutative ring done right in Coq

Grégoire, Mahboubi

2005

Interpret ring expressions as multivariate polynomials much more involved and useful than monoids

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Accelerating verified-compiler development with a verified rewrite engine
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Applying the methodology to get a faster rewrite tactic

Proofs by computation ✨

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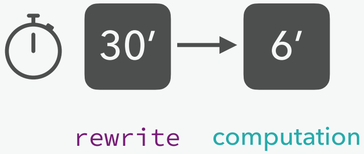
Applying the methodology to get a faster rewrite tactic

Sulfur: a Reflective Tactic for Substitution Simplification
Bouverot-Dupuis, Winterhalter, Stark, Maillard
2026



Limits of computation

(by default)

```
lemma comm : ∀ n m. n + m = m + n
```

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```
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```
induction n
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Limits of computation

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```
0 + m = m + 0
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1

Limits of computation

(by default)

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lemma comm : ∀ n m. n + m = m + n
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```
induction n
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```
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```

1

```
ih : ∀ m. n + m = m + n
```

```
S n + m = m + S n
```

2

Limits of computation

(by default)

```
lemma comm : ∀ n m. n + m = m + n
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0 + m = m + 0
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Limits of computation

(by default)

```
lemma comm : ∀ n m. n + m = m + n
```

```
induction n
```

```
0 + m = m + 0
```

1

```
simpl
```

```
m = m + 0
```

```
ih : ∀ m. n + m = m + n
```

```
S n + m = m + S n
```

2

Limits of computation

(by default)

```
lemma comm : ∀ n m. n + m = m + n
```

```
induction n
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computation is stuck 😔

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Limits of computation

(by default)

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tension between
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with custom computation, one can get the best of both worlds

Controlling and extending computation in ITPs

First extensions of calculus of constructions with rewrite rules

Multiparadigm computational models based on rewriting

Fernández (PhD supervised by Jouannaud)

1993

Modularity of Strong Normalisation and Confluence in the algebraic λ-cube

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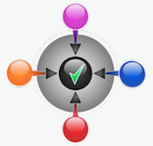

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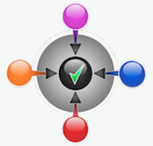

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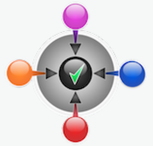

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Cockx, Abel

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Agda

The Rewster: type-preserving rewrite rules for the Coq proof assistant

Leray, Gilbert, Tabareau, Winterhalter

2024

ROCQ

Examples of custom computation rules

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better than going through sorted lists

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```
def nth_exn {A} (l : list A) n : A :=
  match l, n with
  | x :: l, 0 ⇒ x
  | x :: l, S n ⇒ nth_exn l n
  | _, _ ⇒ raise
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no need for the usual error handling with monads (or `option`)

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```
lemma hehe : 0 = 1 :=
  raise
```

☣ the logic is now inconsistent ☣

Equality reflection

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$$\frac{\Gamma \vdash p : u =_A v}{\Gamma \vdash u \equiv v}$$

Maximal extensibility

Equality reflection

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Undecidable type checking

need to rely on heuristics *eg* SMT solvers in F*

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Conservative over ITT + UIP + funext

Extensional concepts in intensional type theory

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Effective translation to ITT

Extensionality in the calculus of constructions

Oury

2005

Eliminating reflection from type theory

Winterhalter, Sozeau, Tabareau

2019

Terribly inefficient!

Let’s go local!

Why locality matters

Example: exceptions

```
symb raise : ∀ {A}. A
rule if raise then t else f ↪ raise
def nth_exn : list A → ℕ → A := …
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Computation rules must be assumed forever

Why locality matters

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Agda ROCQ

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No way to ensure the rules aren't used here
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Rules extend the trusted computing base

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Overall, not very modular
(or type-theoretic)

Our proposal

Prenex quantification over computation rules

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interface Bool
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Equations are verified implicitly

🍰 You can exploit the encoding without having to work directly with it! 😋

Example

Strictification

Autosubst encodes substitutions as functions `ℕ → term` and renamings as `ℕ → ℕ`
to get definitional unitality and associativity of composition

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induces issues with meta-programming
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induces issues with meta-programming
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  assumes
    renaming : Type
    comp : renaming → renaming → renaming
    id : renaming
  where
    comp f id ↠ f
    comp id f ↠ f
    comp (comp f g) h ↠ comp f (comp g h)
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and so on...

Potential example

Strictification

Type theory in type theory using a strictified syntax

Kaposi, Pujet

2025

Categories with families with strict substitution laws

$$(\Pi\ A\ B)[\sigma] = \Pi\ A[\sigma]\ B[\sigma\uparrow]$$

$$(\Pi\ A\ B)[\sigma] \equiv \Pi\ A[\sigma]\ B[\sigma\uparrow]$$

using rewrite rules in Agda

to implement observational equality

Picture from Loïc Pujet

Potential example

Strictification

Type theory in type theory using a strictified syntax

Kaposi, Pujet

2025

Categories with families with strict substitution laws

$$(\Pi\ A\ B)[\sigma] = \Pi\ A[\sigma]\ B[\sigma\uparrow]$$

$$(\Pi\ A\ B)[\sigma] \equiv \Pi\ A[\sigma]\ B[\sigma\uparrow]$$

using rewrite rules in Agda

to implement observational equality

🕵 We need to investigate if our solution could compensate slowdowns

Picture from Loïc Pujet

Example

Hiding implementation details

Hiding implementation details
while retaining computation

```
interface Shift
  assumes
    shift : list ℕ → list ℕ
  where
    shift (x :: l) ↠ S x :: shift l
    shift [] ↠ []
```

```
instance ShiftasMap
  shift := map S
```

This way, `map` never appears in goals out of nowhere

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```
instance ShiftasMap
  shift := map S
```

This way, map never appears in goals out of nowhere

Dream

Use it with Equations (Magin and Sozeau)

```
equations shift : list ℕ → list ℕ
  shift (x :: l) := S x :: shift l
  shift [] := []
```

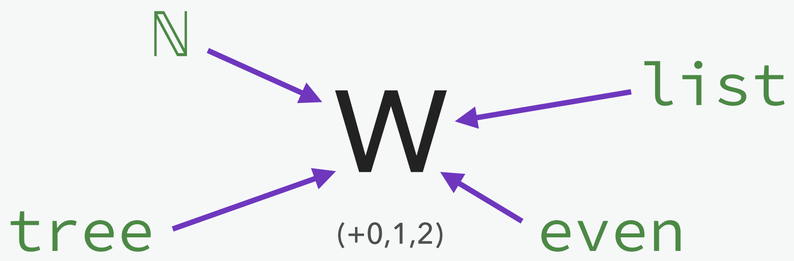
Agda in ROCQ

No more `shift_equation_1` appearing in the goal! 🎉

Example

Inductive types and extensions

Encode features using simpler ones



Why not W?

Hugunin

2021

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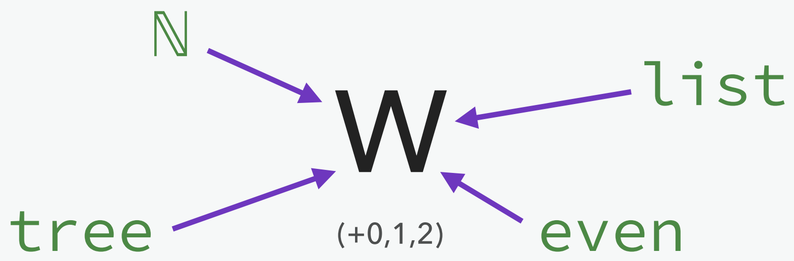

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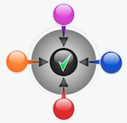

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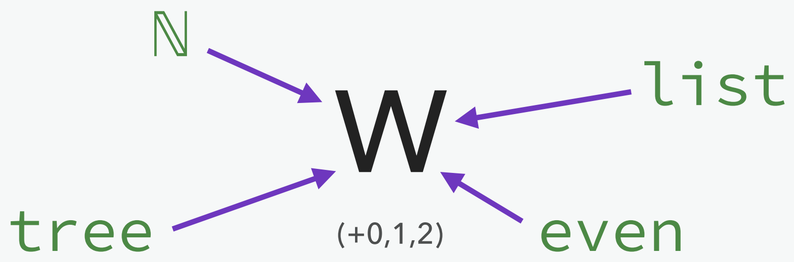

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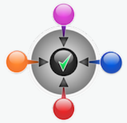

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Use a similar idea to support…

inductive inductive types
inductive recursive types
HITs, QIITs, …

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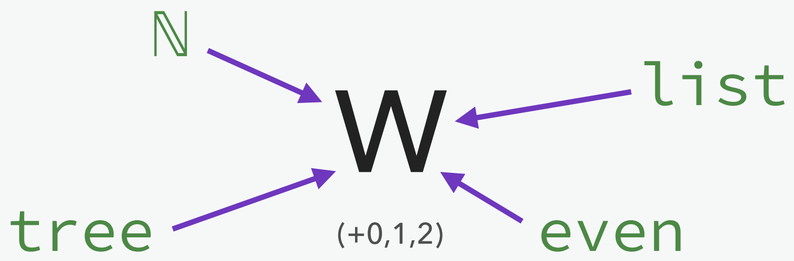

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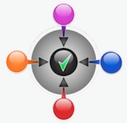


useful for Dedukti which doesn't support inductives

Dream

Use a similar idea to support…

inductive inductive types
inductive recursive types
HITs, QIITs, …

cheaper than extending ROCQ

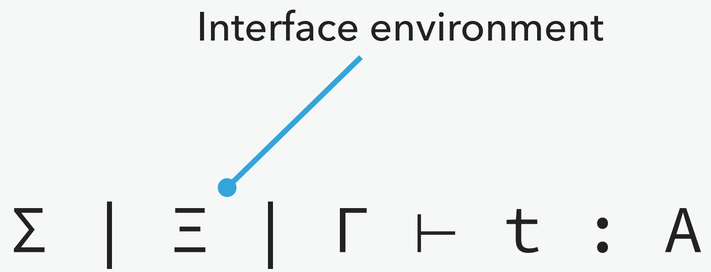
The type theory: LRTT

$$\Sigma \mid \Xi \mid \Gamma \vdash t : A$$

The type theory: LRTT

```
symb x : A
```

```
rule l ↪ r
```



The type theory: LRTT

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Global environment

Interface environment

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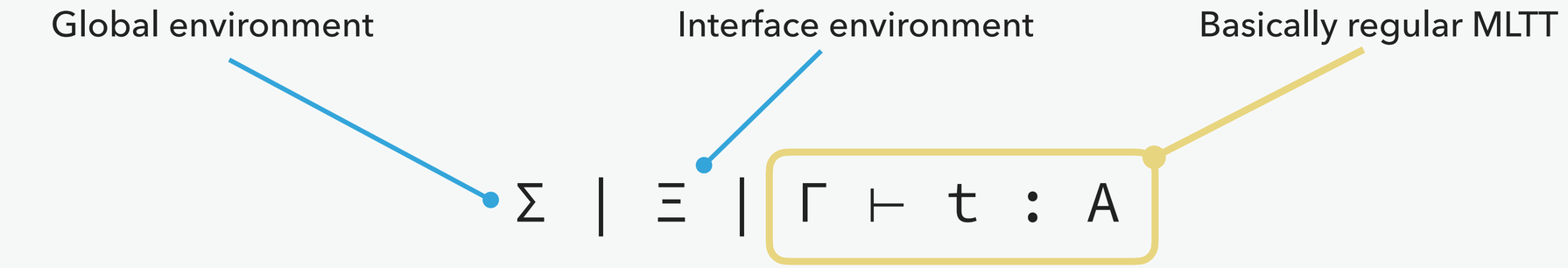

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Basically regular MLTT

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$$\frac{(l \hookrightarrow r) \in \Xi}{\Sigma \mid \Xi \mid \Gamma \vdash l[\sigma] \equiv r[\sigma]}$$

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Meta-theory

mostly usual

Environment weakening (Σ, Ξ, Γ)**, substitution, instantiation, validity**

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Consistency

A given by embedding into a theory
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Consistency

A given by embedding into a theory
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more interesting:

Conservativity over MLTT

```
· | · | · ⊢ A : Type →
Σ | · | · ⊢ t : A →
∃ t'. · | · | · ⊢ t' : A
```

Obtained by **inlining** definitions

Inlining

if $\Sigma \mid \Xi \mid \Gamma \vdash t : A$

then $\cdot \mid [\![\Xi]\!]\langle \kappa \rangle \mid [\![\Gamma]\!]\langle \kappa \rangle \vdash [\![t]\!]\langle \kappa \rangle : [\![A]\!]\langle \kappa \rangle$

where κ interprets the definitions of Σ

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with κ fixed (and abstract):

$$[\![x]\!] := x$$

$$[\![\lambda\ (x : A).\ t]\!] := \lambda\ (x : [\![A]\!]).\ [\![t]\!]$$

$$[\![u\ v]\!] := [\![u]\!]\ [\![v]\!]$$

$$[\![M.x]\!] := M.x$$

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κ is then defined by induction on $\vdash\ \Sigma$ such that

when $(\mathtt{def}\ f\langle\ \Xi'\ \rangle : A := t) \in \Sigma$ we have $\kappa\ f := [\![t]\!]\langle\ \kappa\_rec\ \rangle$

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recursive call ok because t lives in an environment *smaller* than Σ

Inlining

Compared to conservativity, we need full generality here

if $\Sigma \mid \Xi \mid \Gamma \vdash t : A$

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Inlining

if $\Sigma \mid \Xi \mid \Gamma \vdash t : A$ (and $\vdash \Sigma$)

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now we prove

Conservativity over MLTT

$$\cdot \mid \cdot \mid \cdot \vdash A : \mathsf{Type} \to$$
$$\Sigma \mid \cdot \mid \cdot \vdash t : A \to$$
$$\exists\ t'.\ \cdot \mid \cdot \mid \cdot \vdash t' : A$$

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Since A is typed in MLTT
we know it cannot refer to any definition in Σ

$$[\![A]\!]\langle \kappa \rangle = A$$

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Done!

What about type checking?

Checking conversion

We typically need properties like subject reduction and thus injectivity of type formers

so we characterise conversion with (confluent!) reduction

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so we characterise conversion with (confluent!) reduction

Unfolding rule (conversion)

$$\frac{(\mathtt{def}\ f\langle\ \Xi'\ \rangle\ :\ A\ :=\ t)\ \in\ \Sigma \qquad \Sigma\ |\ \Xi\ |\ \Gamma\ \vdash\ \xi\ :\ \Xi'}{\Sigma\ |\ \Xi\ |\ \Gamma\ \vdash\ f\langle\xi\rangle\ \equiv\ t\xi}$$

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no `let` so
no local env

strip away side conditions to make confluence easier

Checking conversion

Computation rule (conversion)

$$\frac{(l \twoheadrightarrow r) \in \Xi}{\Sigma \mid \Xi \mid \Gamma \vdash l[\sigma] \equiv r[\sigma]}$$

Computation rule (reduction)

$$\frac{(l \twoheadrightarrow r) \in \Xi}{\Sigma \mid \Xi \vdash l[\sigma] \twoheadrightarrow r[\sigma]}$$

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the rest is similar...

Checking conversion

conversion

$$\Sigma \mid \Xi \mid \Gamma \vdash u \equiv v$$

one-step reduction

$$\Sigma \mid \Xi \vdash u \rightsquigarrow v$$

Checking conversion

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$$\Sigma \mid \Xi \mid \Gamma \vdash u \equiv v$$

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joinability ($\rightsquigarrow^{*} \cdot {}^{*}\!\!\leftsquigarrow$)

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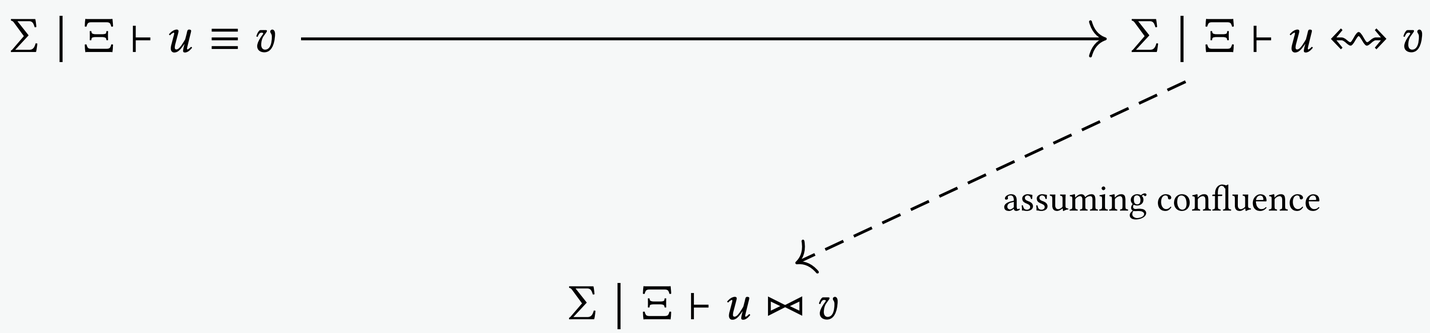

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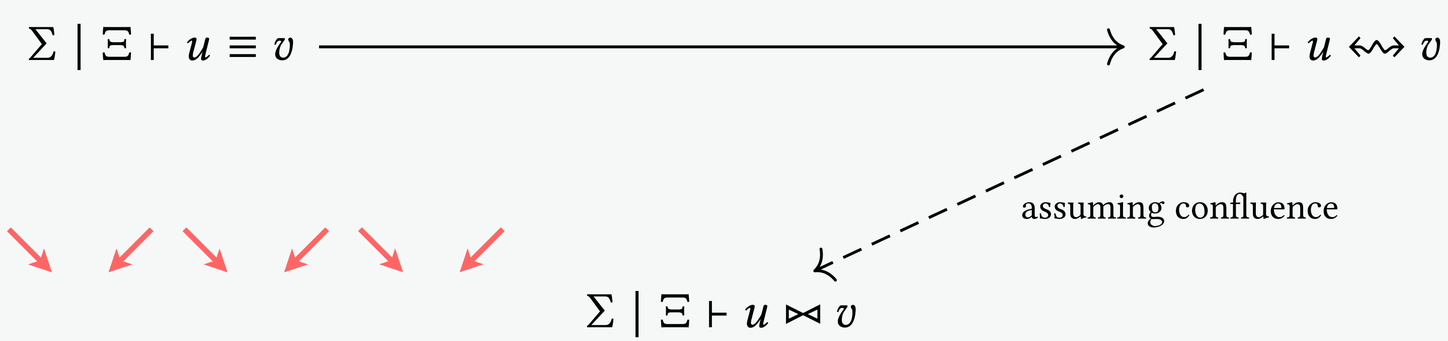

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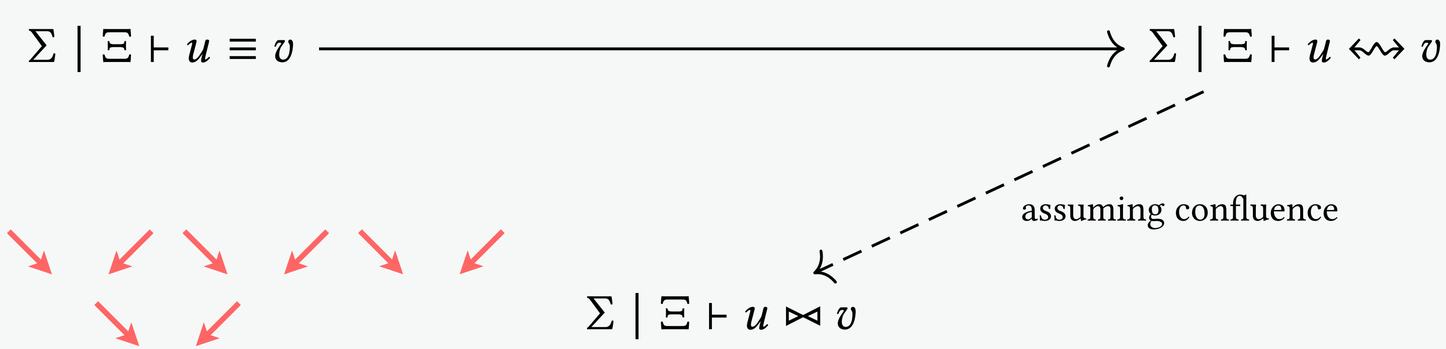

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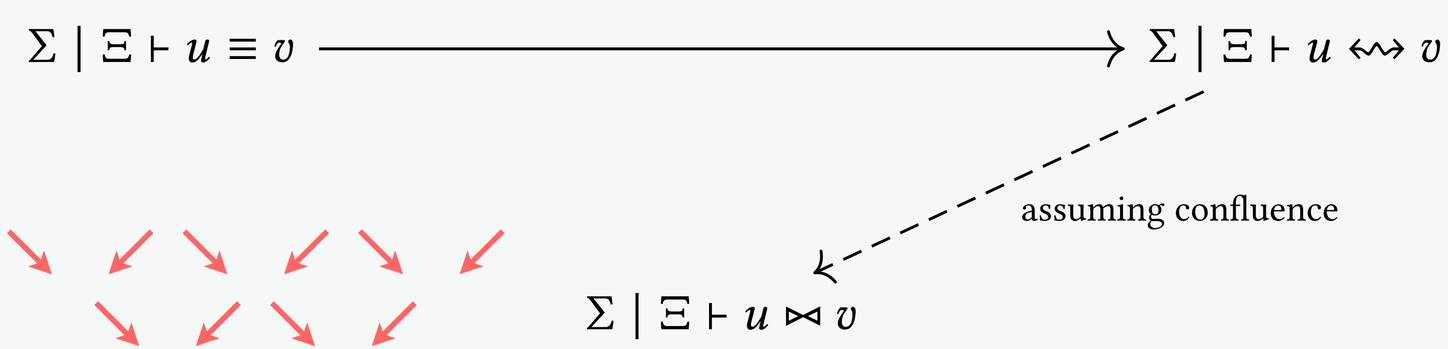

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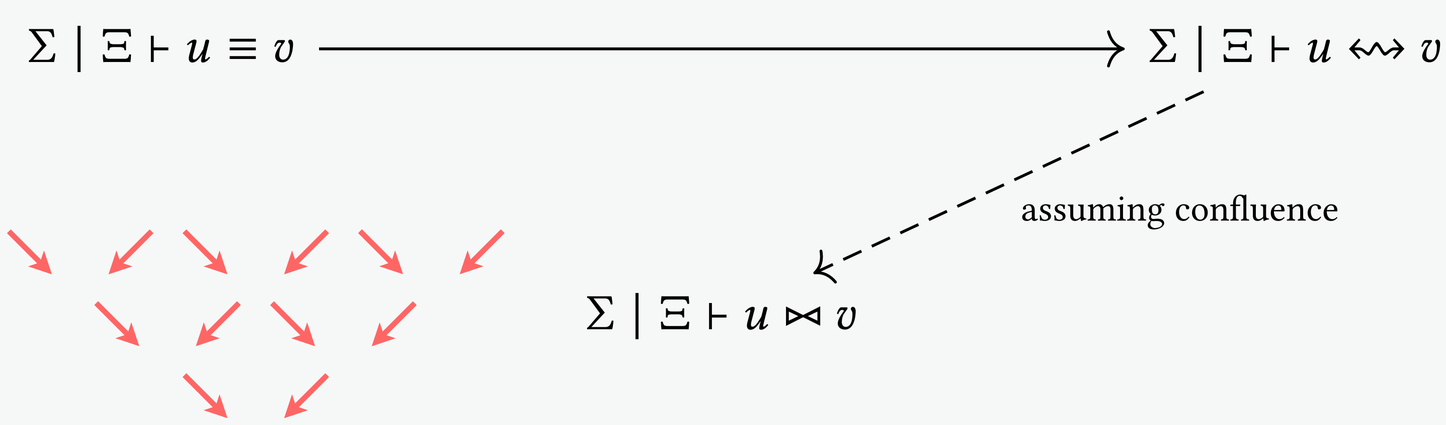


Checking conversion

conversion

$$\Sigma \mid \Xi \mid \Gamma \vdash u \equiv v$$

one-step reduction

$$\Sigma \mid \Xi \vdash u \rightsquigarrow v$$

many-step reduction

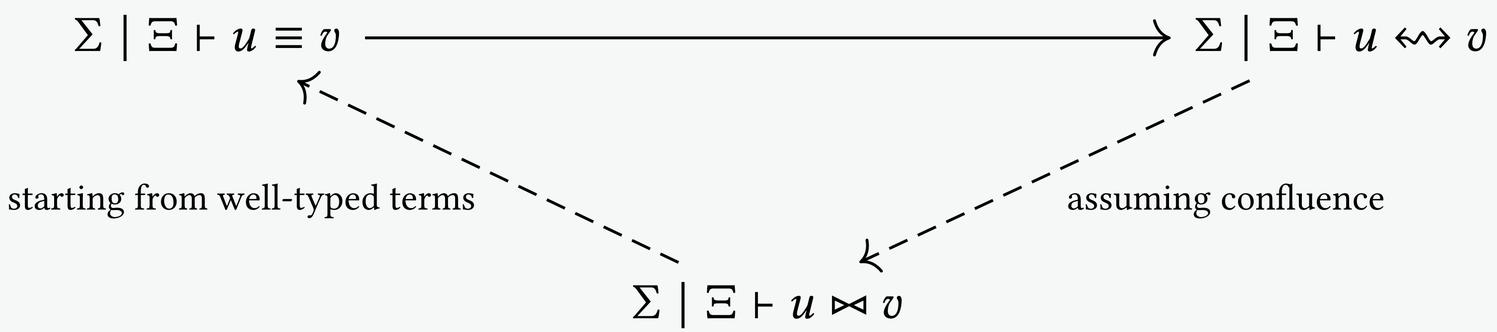
$$\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$$

congruence closure of reduction

$$\Sigma \mid \Xi \vdash u \leftrightsquigarrow v$$

joinability ($\rightsquigarrow^* \cdot {}^* \leftsquigarrow$)

$$\Sigma \mid \Xi \vdash u \bowtie v$$



Checking conversion

conversion

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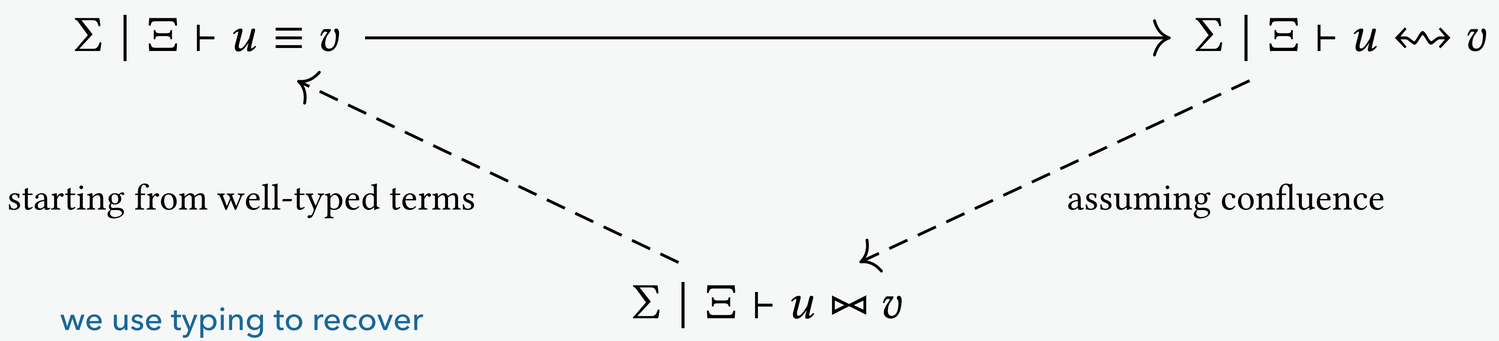

we use typing to recover
the missing side conditions!

Checking conversion

conversion

$$\Sigma \mid \Xi \mid \Gamma \vdash u \equiv v$$

one-step reduction

$$\Sigma \mid \Xi \vdash u \rightsquigarrow v$$

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$$\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$$

congruence closure of reduction

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joinability ($\rightsquigarrow^* \cdot {}^* \!\!\leftsquigarrow$)

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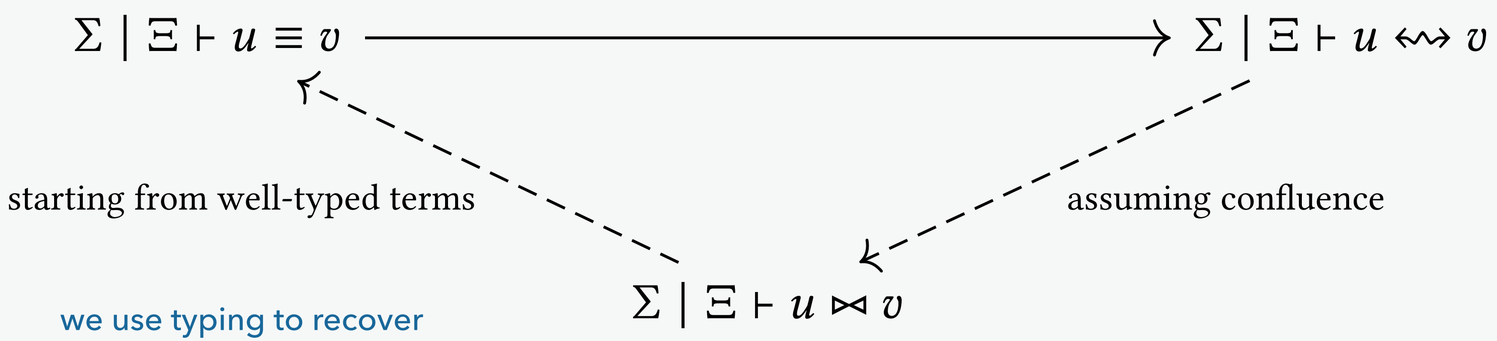

we use typing to recover
the missing side conditions!

easy?

From joinability to conversion (failed attempt)

conversion

$$\Sigma \mid \Xi \mid \Gamma \vdash u \equiv v$$

one-step reduction

$$\Sigma \mid \Xi \vdash u \rightsquigarrow v$$

many-step reduction

$$\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$$

congruence closure of reduction

$$\Sigma \mid \Xi \vdash u \leftrightsquigarrow v$$

joinability ($\rightsquigarrow^* \cdot {}^* \leftsquigarrow$)

$$\Sigma \mid \Xi \vdash u \bowtie v$$

Assume that if $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ and $\Sigma \mid \Xi \mid \Gamma \vdash u : A$ then $\Sigma \mid \Xi \vdash u \equiv v$

From joinability to conversion (failed attempt)

conversion

$$\Sigma \mid \Xi \mid \Gamma \vdash u \equiv v$$

one-step reduction

$$\Sigma \mid \Xi \vdash u \rightsquigarrow v$$

many-step reduction

$$\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$$

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Assume that if $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ and $\Sigma \mid \Xi \mid \Gamma \vdash u : A$ then $\Sigma \mid \Xi \vdash u \equiv v$

By induction on $\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$ such that $\Sigma \mid \Xi \mid \Gamma \vdash u : A$ we show $\Sigma \mid \Xi \vdash u \equiv v$

From joinability to conversion (failed attempt)

conversion

$$\Sigma \mid \Xi \mid \Gamma \vdash u \equiv v$$

one-step reduction

$$\Sigma \mid \Xi \vdash u \rightsquigarrow v$$

many-step reduction

$$\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$$

congruence closure of reduction

$$\Sigma \mid \Xi \vdash u \leftrightsquigarrow v$$

joinability ($\rightsquigarrow^* \cdot {}^* \!\!\leftsquigarrow$)

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Assume that if $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ and $\Sigma \mid \Xi \mid \Gamma \vdash u : A$ then $\Sigma \mid \Xi \vdash u \equiv v$

By induction on $\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$ such that $\Sigma \mid \Xi \mid \Gamma \vdash u : A$ we show $\Sigma \mid \Xi \vdash u \equiv v$

We have $\Sigma \mid \Xi \vdash u \rightsquigarrow w$ and $\Sigma \mid \Xi \vdash w \rightsquigarrow^* v$

From joinability to conversion (failed attempt)

conversion

$$\Sigma \mid \Xi \mid \Gamma \vdash u \equiv v$$

one-step reduction

$$\Sigma \mid \Xi \vdash u \rightsquigarrow v$$

many-step reduction

$$\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$$

congruence closure of reduction

$$\Sigma \mid \Xi \vdash u \leftrightsquigarrow v$$

joinability ($\rightsquigarrow^* \cdot {}^* \!\!\leftsquigarrow$)

$$\Sigma \mid \Xi \vdash u \bowtie v$$

Assume that if $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ and $\Sigma \mid \Xi \mid \Gamma \vdash u : A$ then $\Sigma \mid \Xi \vdash u \equiv v$

By induction on $\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$ such that $\Sigma \mid \Xi \mid \Gamma \vdash u : A$ we show $\Sigma \mid \Xi \vdash u \equiv v$

We have $\Sigma \mid \Xi \vdash u \rightsquigarrow w$ and $\Sigma \mid \Xi \vdash w \rightsquigarrow^* v$ thus $\Sigma \mid \Xi \vdash u \equiv w$

From joinability to conversion (failed attempt)

conversion

$$\Sigma \mid \Xi \mid \Gamma \vdash u \equiv v$$

one-step reduction

$$\Sigma \mid \Xi \vdash u \rightsquigarrow v$$

many-step reduction

$$\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$$

congruence closure of reduction

$$\Sigma \mid \Xi \vdash u \leftrightsquigarrow v$$

joinability ($\rightsquigarrow^*\cdot{}^*\!\!\leftsquigarrow$)

$$\Sigma \mid \Xi \vdash u \bowtie v$$

Assume that if $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ and $\Sigma \mid \Xi \mid \Gamma \vdash u : A$ then $\Sigma \mid \Xi \vdash u \equiv v$

By induction on $\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$ such that $\Sigma \mid \Xi \mid \Gamma \vdash u : A$ we show $\Sigma \mid \Xi \vdash u \equiv v$

We have $\Sigma \mid \Xi \vdash u \rightsquigarrow w$ and $\Sigma \mid \Xi \vdash w \rightsquigarrow^* v$ thus $\Sigma \mid \Xi \vdash u \equiv w$

To apply IH and conclude with $\Sigma \mid \Xi \vdash w \equiv v$ we need $\Sigma \mid \Xi \mid \Gamma \vdash w : A$

From joinability to conversion (failed attempt)

conversion

$$\Sigma \mid \Xi \mid \Gamma \vdash u \equiv v$$

one-step reduction

$$\Sigma \mid \Xi \vdash u \rightsquigarrow v$$

many-step reduction

$$\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$$

congruence closure of reduction

$$\Sigma \mid \Xi \vdash u \leftrightsquigarrow v$$

joinability ($\rightsquigarrow^* \cdot {}^* \!\!\leftsquigarrow$)

$$\Sigma \mid \Xi \vdash u \bowtie v$$

Assume that if $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ and $\Sigma \mid \Xi \mid \Gamma \vdash u : A$ then $\Sigma \mid \Xi \vdash u \equiv v$

By induction on $\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$ such that $\Sigma \mid \Xi \mid \Gamma \vdash u : A$ we show $\Sigma \mid \Xi \vdash u \equiv v$

We have $\Sigma \mid \Xi \vdash u \rightsquigarrow w$ and $\Sigma \mid \Xi \vdash w \rightsquigarrow^* v$ thus $\Sigma \mid \Xi \vdash u \equiv w$

To apply IH and conclude with $\Sigma \mid \Xi \vdash w \equiv v$ we need $\Sigma \mid \Xi \mid \Gamma \vdash w : A$ meaning subject reduction!

Subject reduction (failed attempt)

Subject reduction

If $\Sigma \mid \Xi \mid \Gamma \vdash u : T$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ then $\Sigma \mid \Xi \mid \Gamma \vdash v : T$

Subject reduction (failed attempt)

Subject reduction

If $\Sigma \mid \Xi \mid \Gamma \vdash u : T$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ then $\Sigma \mid \Xi \mid \Gamma \vdash v : T$

By induction on reduction, β case:

$$\Sigma \mid \Xi \vdash (\lambda\ (x : A).\ t)\ u \rightsquigarrow t[x := u]$$

Subject reduction (failed attempt)

Subject reduction

If $\Sigma \mid \Xi \mid \Gamma \vdash u : T$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ then $\Sigma \mid \Xi \mid \Gamma \vdash v : T$

By induction on reduction, β case:

$$\Sigma \mid \Xi \vdash (\lambda\ (x : A).\ t)\ u \rightsquigarrow t[x := u]$$

inversion on typing on $\Sigma \mid \Xi \mid \Gamma \vdash (\lambda\ (x : A).\ t)\ u : T$ gives

Subject reduction (failed attempt)

Subject reduction

If `Σ | Ξ | Γ ⊢ u : T` and `Σ | Ξ ⊢ u ⇝ v` then `Σ | Ξ | Γ ⊢ v : T`

By induction on reduction, β case:

```
Σ | Ξ ⊢ (λ (x : A). t) u ⇝ t[x := u]
```

inversion on typing on `Σ | Ξ | Γ ⊢ (λ (x : A). t) u : T` gives

```
Σ | Ξ | Γ ⊢ λ (x : A). t : Π (x : C). D
```

Subject reduction (failed attempt)

Subject reduction

If $\Sigma \mid \Xi \mid \Gamma \vdash u : T$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ then $\Sigma \mid \Xi \mid \Gamma \vdash v : T$

By induction on reduction, β case:

$$\Sigma \mid \Xi \vdash (\lambda\ (x : A).\ t)\ u \rightsquigarrow t[x := u]$$

inversion on typing on $\Sigma \mid \Xi \mid \Gamma \vdash (\lambda\ (x : A).\ t)\ u : T$ gives

$\Sigma \mid \Xi \mid \Gamma \vdash \lambda\ (x : A).\ t : \Pi\ (x : C).\ D$ $\Sigma \mid \Xi \mid \Gamma \vdash u : C$

Subject reduction (failed attempt)

Subject reduction

If $\Sigma \mid \Xi \mid \Gamma \vdash u : T$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ then $\Sigma \mid \Xi \mid \Gamma \vdash v : T$

By induction on reduction, β case:

$$\Sigma \mid \Xi \vdash (\lambda\ (x : A).\ t)\ u \rightsquigarrow t[x := u]$$

inversion on typing on $\Sigma \mid \Xi \mid \Gamma \vdash (\lambda\ (x : A).\ t)\ u : T$ gives

$\Sigma \mid \Xi \mid \Gamma \vdash \lambda\ (x : A).\ t : \Pi\ (x : C).\ D$ $\quad$ $\Sigma \mid \Xi \mid \Gamma \vdash u : C$ $\quad$ $\Sigma \mid \Xi \mid \Gamma \vdash D[x := u] \equiv T$

Subject reduction (failed attempt)

Subject reduction

If $\Sigma \mid \Xi \mid \Gamma \vdash u : T$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ then $\Sigma \mid \Xi \mid \Gamma \vdash v : T$

By induction on reduction, β case:

$$\Sigma \mid \Xi \vdash (\lambda\ (x : A).\ t)\ u \rightsquigarrow t[x := u]$$

inversion on typing on $\Sigma \mid \Xi \mid \Gamma \vdash (\lambda\ (x : A).\ t)\ u : T$ gives

$\Sigma \mid \Xi \mid \Gamma \vdash \lambda\ (x : A).\ t : \Pi\ (x : C).\ D$ $\quad \Sigma \mid \Xi \mid \Gamma \vdash u : C$ $\quad \Sigma \mid \Xi \mid \Gamma \vdash D[x := u] \equiv T$

inversion again

Subject reduction (failed attempt)

Subject reduction

If $\Sigma \mid \Xi \mid \Gamma \vdash u : T$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ then $\Sigma \mid \Xi \mid \Gamma \vdash v : T$

By induction on reduction, β case:

$$\Sigma \mid \Xi \vdash (\lambda\ (x : A).\ t)\ u \rightsquigarrow t[x := u]$$

inversion on typing on $\Sigma \mid \Xi \mid \Gamma \vdash (\lambda\ (x : A).\ t)\ u : T$ gives

$\Sigma \mid \Xi \mid \Gamma \vdash \lambda\ (x : A).\ t : \Pi\ (x : C).\ D$ $\quad \Sigma \mid \Xi \mid \Gamma \vdash u : C$ $\quad \Sigma \mid \Xi \mid \Gamma \vdash D[x := u] \equiv T$

inversion again $\Sigma \mid \Xi \mid \Gamma, x : A \vdash t : B$

Subject reduction (failed attempt)

Subject reduction

If $\Sigma \mid \Xi \mid \Gamma \vdash u : T$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ then $\Sigma \mid \Xi \mid \Gamma \vdash v : T$

By induction on reduction, β case:

$$\Sigma \mid \Xi \vdash (\lambda\ (x : A).\ t)\ u \rightsquigarrow t[x := u]$$

inversion on typing on $\Sigma \mid \Xi \mid \Gamma \vdash (\lambda\ (x : A).\ t)\ u : T$ gives

$\Sigma \mid \Xi \mid \Gamma \vdash \lambda\ (x : A).\ t : \Pi\ (x : C).\ D$ $\quad \Sigma \mid \Xi \mid \Gamma \vdash u : C$ $\quad \Sigma \mid \Xi \mid \Gamma \vdash D[x := u] \equiv T$

inversion again $\Sigma \mid \Xi \mid \Gamma, x : A \vdash t : B$ $\quad \Sigma \mid \Xi \mid \Gamma \vdash \Pi\ (x : A).\ B \equiv \Pi\ (x : C).\ D$

Subject reduction (failed attempt)

Subject reduction

If $\Sigma \mid \Xi \mid \Gamma \vdash u : T$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ then $\Sigma \mid \Xi \mid \Gamma \vdash v : T$

By induction on reduction, β case:

$$\Sigma \mid \Xi \vdash (\lambda\ (x : A).\ t)\ u \rightsquigarrow t[x := u]$$

inversion on typing on $\Sigma \mid \Xi \mid \Gamma \vdash (\lambda\ (x : A).\ t)\ u : T$ gives

$\Sigma \mid \Xi \mid \Gamma \vdash \lambda\ (x : A).\ t : \Pi\ (x : C).\ D$ $\quad \Sigma \mid \Xi \mid \Gamma \vdash u : C$ $\quad \Sigma \mid \Xi \mid \Gamma \vdash D[x := u] \equiv T$

inversion again $\Sigma \mid \Xi \mid \Gamma, x : A \vdash t : B$ $\quad \Sigma \mid \Xi \mid \Gamma \vdash \Pi\ (x : A).\ B \equiv \Pi\ (x : C).\ D$

to conclude $\Sigma \mid \Xi \mid \Gamma \vdash t[x := u] : D[x := u]$ we want

$\Sigma \mid \Xi \mid \Gamma \vdash A \equiv C$ $\quad \Sigma \mid \Xi \mid \Gamma \vdash B[x := u] \equiv D[x := u]$

Subject reduction (failed attempt)

Subject reduction

If $\Sigma \mid \Xi \mid \Gamma \vdash u : T$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ then $\Sigma \mid \Xi \mid \Gamma \vdash v : T$

By induction on reduction, β case:

$$\Sigma \mid \Xi \vdash (\lambda\ (x : A).\ t)\ u \rightsquigarrow t[x := u]$$

inversion on typing on $\Sigma \mid \Xi \mid \Gamma \vdash (\lambda\ (x : A).\ t)\ u : T$ gives

$\Sigma \mid \Xi \mid \Gamma \vdash \lambda\ (x : A).\ t : \Pi\ (x : C).\ D$ $\quad$ $\Sigma \mid \Xi \mid \Gamma \vdash u : C$ $\quad$ $\Sigma \mid \Xi \mid \Gamma \vdash D[x := u] \equiv T$

inversion again $\Sigma \mid \Xi \mid \Gamma, x : A \vdash t : B$ $\quad$ $\Sigma \mid \Xi \mid \Gamma \vdash \Pi\ (x : A).\ B \equiv \Pi\ (x : C).\ D$

injectivity of Π

to conclude $\Sigma \mid \Xi \mid \Gamma \vdash t[x := u] : D[x := u]$ we want

$\Sigma \mid \Xi \mid \Gamma \vdash A \equiv C$ $\quad$ $\Sigma \mid \Xi \mid \Gamma \vdash B[x := u] \equiv D[x := u]$

Injectivity of Π (failed attempt)

Injectivity of Π

If `Σ | Ξ | Γ ⊢ Π (x : A). B ≡ Π (x : C). D` then
`Σ | Ξ | Γ ⊢ A ≡ C` and `Σ | Ξ | Γ, x : A ⊢ B ≡ D`

Injectivity of Π (failed attempt)

Injectivity of Π

If $\Sigma \mid \Xi \mid \Gamma \vdash \Pi\ (x : A).\ B \equiv \Pi\ (x : C).\ D$ then
$\Sigma \mid \Xi \mid \Gamma \vdash A \equiv C$ and $\Sigma \mid \Xi \mid \Gamma, x : A \vdash B \equiv D$

assuming confluence $\Sigma \mid \Xi \vdash \Pi\ (x : A).\ B \bowtie \Pi\ (x : C).\ D$

Injectivity of Π (failed attempt)

Injectivity of Π

If $\Sigma \mid \Xi \mid \Gamma \vdash \Pi\ (x : A).\ B \equiv \Pi\ (x : C).\ D$ then
$\Sigma \mid \Xi \mid \Gamma \vdash A \equiv C$ and $\Sigma \mid \Xi \mid \Gamma, x : A \vdash B \equiv D$

assuming confluence $\Sigma \mid \Xi \vdash \Pi\ (x : A).\ B \bowtie \Pi\ (x : C).\ D$

since no reduction rules on Π besides congruence

$\Sigma \mid \Xi \vdash A \bowtie C$ $\Sigma \mid \Xi \vdash B \bowtie D$

Injectivity of Π (failed attempt)

Injectivity of Π

If $\Sigma \mid \Xi \mid \Gamma \vdash \Pi\ (x : A).\ B \equiv \Pi\ (x : C).\ D$ then
$\Sigma \mid \Xi \mid \Gamma \vdash A \equiv C$ and $\Sigma \mid \Xi \mid \Gamma, x : A \vdash B \equiv D$

assuming confluence $\Sigma \mid \Xi \vdash \Pi\ (x : A).\ B \bowtie \Pi\ (x : C).\ D$

since no reduction rules on Π besides congruence

$\Sigma \mid \Xi \vdash A \bowtie C$ $\Sigma \mid \Xi \vdash B \bowtie D$

we conclude by going back to conversion 🎉

Injectivity of Π (failed attempt)

Injectivity of Π

If $\Sigma \mid \Xi \mid \Gamma \vdash \Pi\ (x : A).\ B \equiv \Pi\ (x : C).\ D$ then
$\Sigma \mid \Xi \mid \Gamma \vdash A \equiv C$ and $\Sigma \mid \Xi \mid \Gamma, x : A \vdash B \equiv D$

assuming confluence $\Sigma \mid \Xi \vdash \Pi\ (x : A).\ B \bowtie \Pi\ (x : C).\ D$

since no reduction rules on Π besides congruence ⚠️ needs restriction on Ξ

$\Sigma \mid \Xi \vdash A \bowtie C$ $\Sigma \mid \Xi \vdash B \bowtie D$

we conclude by going back to conversion 🎉

Injectivity of Π (failed attempt)

Injectivity of Π

If $\Sigma \mid \Xi \mid \Gamma \vdash \Pi\ (x : A).\ B \equiv \Pi\ (x : C).\ D$ then
$\Sigma \mid \Xi \mid \Gamma \vdash A \equiv C$ and $\Sigma \mid \Xi \mid \Gamma, x : A \vdash B \equiv D$

assuming confluence $\Sigma \mid \Xi \vdash \Pi\ (x : A).\ B \bowtie \Pi\ (x : C).\ D$

since no reduction rules on Π besides congruence

⚠️ needs restriction on Ξ

$\Sigma \mid \Xi \vdash A \bowtie C$ $\Sigma \mid \Xi \vdash B \bowtie D$

~~we conclude by going back to conversion~~ 🎉

we need the joinability to conversion
theorem that we're trying to prove!

Injectivity of Π (failed attempt)

Injectivity of Π

If $\Sigma \mid \Xi \mid \Gamma \vdash \Pi\ (x : A).\ B \equiv \Pi\ (x : C).\ D$ then
$\Sigma \mid \Xi \mid \Gamma \vdash A \equiv C$ and $\Sigma \mid \Xi \mid \Gamma, x : A \vdash B \equiv D$

assuming confluence $\Sigma \mid \Xi \vdash \Pi\ (x : A).\ B \bowtie \Pi\ (x : C).\ D$

since no reduction rules on Π besides congruence

⚠️ needs restriction on Ξ

$\Sigma \mid \Xi \vdash A \bowtie C$ $\Sigma \mid \Xi \vdash B \bowtie D$

~~we conclude by going back to conversion~~ 🎉

we need the joinability to conversion
theorem that we're trying to prove!

we're going to use a property weaker than typing to break the loop

From joinability to conversion (for real this time)

$$\Sigma \mid \Xi \ltimes t$$

instances that appear in `t`
respect the equations of the interface they correspond to

$$\frac{(\mathtt{def}\ f\langle\ \Xi'\ \rangle\ :\ A\ :=\ t) \in \Sigma \qquad \Sigma \mid \Xi \vdash \xi \vDash \Xi' \qquad \forall x.\ \Sigma \mid \Xi \ltimes \xi(x)}{\Sigma \mid \Xi \ltimes f\langle\xi\rangle \equiv t\xi}$$

From joinability to conversion (for real this time)

$$\Sigma \mid \Xi \ltimes t$$

instances that appear in `t`
respect the equations of the interface they correspond to

$$\frac{(\mathtt{def}\ f\langle\ \Xi'\ \rangle\ :\ A\ :=\ t)\ \in\ \Sigma \qquad \Sigma \mid \Xi \vdash \xi \vDash \Xi' \qquad \forall x.\ \Sigma \mid \Xi \ltimes \xi(x)}{\Sigma \mid \Xi \ltimes f\langle\xi\rangle \equiv t\xi}$$

Lemma If $\Sigma \mid \Xi \mid \Gamma \vdash u : T$ then $\Sigma \mid \Xi \ltimes u$

From joinability to conversion (for real this time)

$$\Sigma \mid \Xi \ltimes t$$

instances that appear in t
respect the equations of the interface they correspond to

$$\frac{(\mathsf{def}\ f\langle\ \Xi'\ \rangle : A := t) \in \Sigma \qquad \Sigma \mid \Xi \vdash \xi \vDash \Xi' \qquad \forall x.\ \Sigma \mid \Xi \ltimes \xi(x)}{\Sigma \mid \Xi \ltimes f\langle\xi\rangle \equiv t\xi}$$

Lemma If $\Sigma \mid \Xi \mid \Gamma \vdash u : T$ then $\Sigma \mid \Xi \ltimes u$

Lemma If $\Sigma \mid \Xi \ltimes u$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ then $\Sigma \mid \Xi \ltimes v$

From joinability to conversion (for real this time)

$$\Sigma \mid \Xi \ltimes t$$

instances that appear in t
respect the equations of the interface they correspond to

$$\frac{(\mathsf{def}\ f\langle\ \Xi'\ \rangle : A := t) \in \Sigma \qquad \Sigma \mid \Xi \vdash \xi \vDash \Xi' \qquad \forall x.\ \Sigma \mid \Xi \ltimes \xi(x)}{\Sigma \mid \Xi \ltimes f\langle\xi\rangle \equiv t\xi}$$

Lemma If $\Sigma \mid \Xi \mid \Gamma \vdash u : T$ then $\Sigma \mid \Xi \ltimes u$

Lemma If $\Sigma \mid \Xi \ltimes u$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ then $\Sigma \mid \Xi \ltimes v$

Lemma If $\Sigma \mid \Xi \ltimes u$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$ then $\Sigma \mid \Xi \vdash u \equiv v$

From joinability to conversion (for real this time)

$$\Sigma \mid \Xi \ltimes t$$

instances that appear in t
respect the equations of the interface they correspond to

$$\frac{(\mathtt{def}\ f\langle\ \Xi'\ \rangle : A := t) \in \Sigma \qquad \Sigma \mid \Xi \vdash \xi \vDash \Xi' \qquad \forall x.\ \Sigma \mid \Xi \ltimes \xi(x)}{\Sigma \mid \Xi \ltimes f\langle\xi\rangle \equiv t\xi}$$

Lemma If $\Sigma \mid \Xi \mid \Gamma \vdash u : T$ then $\Sigma \mid \Xi \ltimes u$

Lemma If $\Sigma \mid \Xi \ltimes u$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ then $\Sigma \mid \Xi \ltimes v$

Lemma If $\Sigma \mid \Xi \ltimes u$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$ then $\Sigma \mid \Xi \vdash u \equiv v$

Theorem If $\Sigma \mid \Xi \ltimes u$ and $\Sigma \mid \Xi \ltimes v$ then conversion and joinability are equivalent for u and v

From joinability to conversion (for real this time)

$$\Sigma \mid \Xi \ltimes t$$

instances that appear in t
respect the equations of the interface they correspond to

$$\frac{(\texttt{def}\ f\langle\ \Xi'\ \rangle : A := t) \in \Sigma \qquad \Sigma \mid \Xi \vdash \xi \vDash \Xi' \qquad \forall x.\ \Sigma \mid \Xi \ltimes \xi(x)}{\Sigma \mid \Xi \ltimes f\langle\xi\rangle \equiv t\xi}$$

Lemma If $\Sigma \mid \Xi \mid \Gamma \vdash u : T$ then $\Sigma \mid \Xi \ltimes u$

Lemma If $\Sigma \mid \Xi \ltimes u$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow v$ then $\Sigma \mid \Xi \ltimes v$

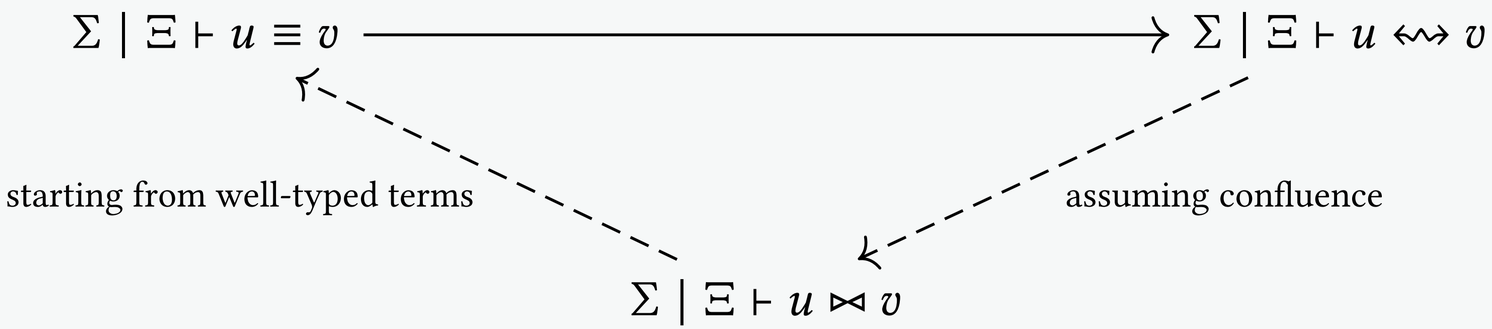
Lemma If $\Sigma \mid \Xi \ltimes u$ and $\Sigma \mid \Xi \vdash u \rightsquigarrow^* v$ then $\Sigma \mid \Xi \vdash u \equiv v$

assuming rules preserve $\ltimes$

Theorem If $\Sigma \mid \Xi \ltimes u$ and $\Sigma \mid \Xi \ltimes v$ then conversion and joinability are equivalent for u and v

Checking conversion

Characterising conversion with reduction



Injectivity of Π

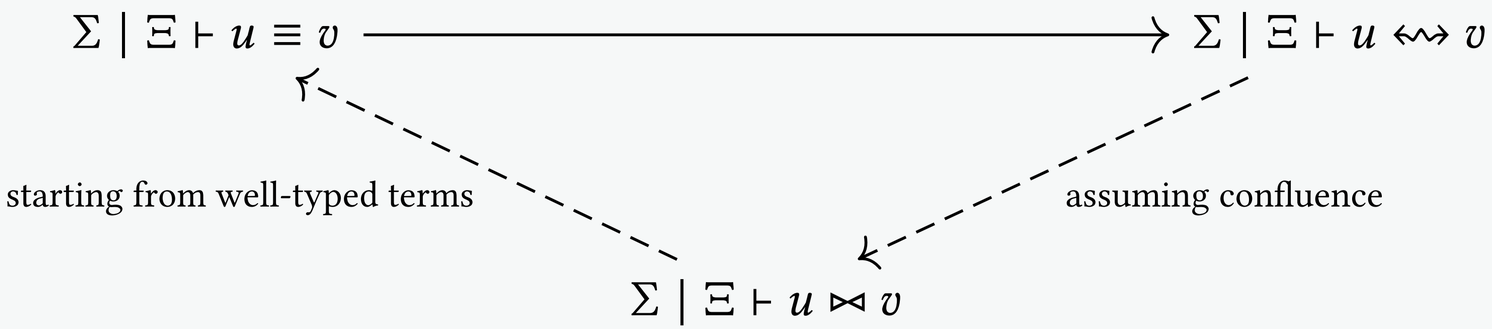
If `Σ | Ξ | Γ ⊢ Π (x : A). B ≡ Π (x : C). D` then
`Σ | Ξ | Γ ⊢ A ≡ C` and `Σ | Ξ | Γ, x : A ⊢ B ≡ D`

Subject reduction

If `Σ | Ξ | Γ ⊢ u : T` and `Σ | Ξ ⊢ u ⇝ v` then `Σ | Ξ | Γ ⊢ v : T`

Checking conversion

Characterising conversion with reduction



Injectivity of Π

If `Σ | Ξ | Γ ⊢ Π (x : A). B ≡ Π (x : C). D` then
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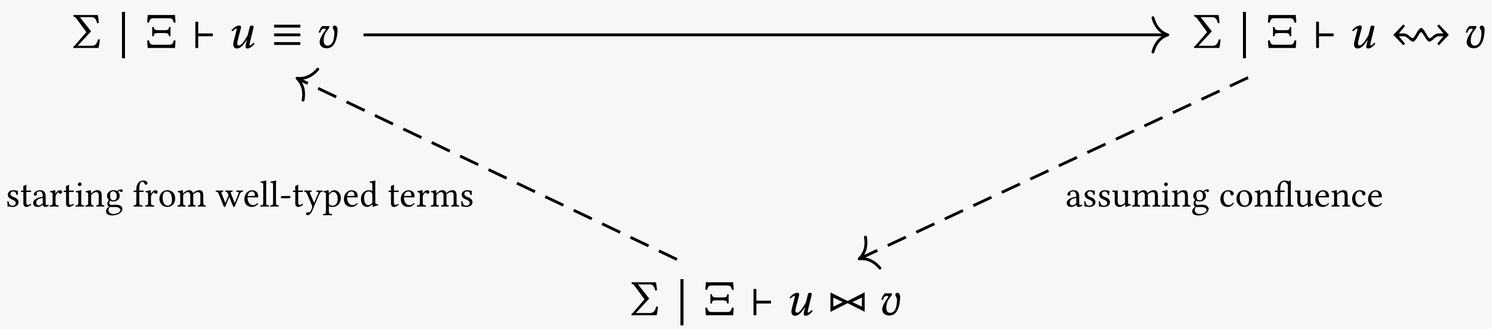

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assuming rules preserve $\bowtie$ and typing + confluence

it holds with no rules at top-level (so you can wreak havoc locally without creating trouble globally)

How to check these conditions

Modular confluence criterion

The Taming of the Rew: A type theory with computational assumptions

Cockx, Tabareau, Winterhalter

2021

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Modular type safety conditions

Type safety of rewrite rules in dependent types

Blanqui

2020

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left to the user
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+ many tools developed by the rewriting community!

ROCQ prototype

on top of

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ROCQ prototype

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2024

using the module system for interfaces and instances
and functors to quantify over interfaces

```
Module Type Plus.
  Symbol plus : ℕ → ℕ → ℕ.
  Rewrite Rules plus_r :=
  | plus 0 ?n => ?n
  | plus (S ?m) ?n => S (plus ?m ?n).
End Plus.
```

ROCQ prototype

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Module UsePlus (P : Plus).
  Import P.

  Compute (plus 2 3). (* 5 *)

End UsePlus.
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ROCQ prototype

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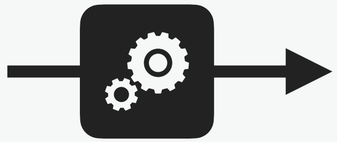
End UsePlus.
```

```
Module Import PlusImpl : Plus.
  Definition plus := Nat.add.
End PlusImpl.
```

Conclusion

Conservative extension of MLTT with **local computation**

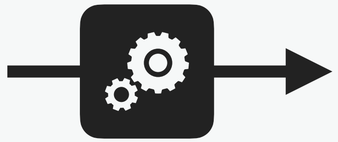
Conclusion



hide implementation details

Conservative extension of MLTT with **local computation**

Conclusion



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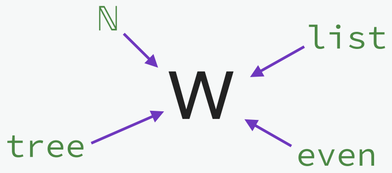
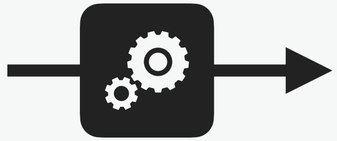


encode features

Conservative extension of MLTT with **local computation**

Conclusion



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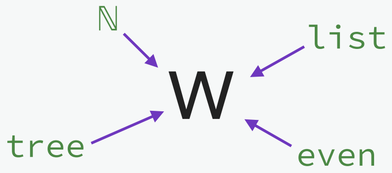

encode features

contained extensions (safer)

Conservative extension of MLTT with **local computation**

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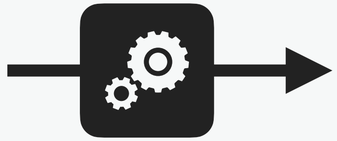

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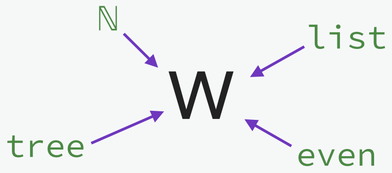

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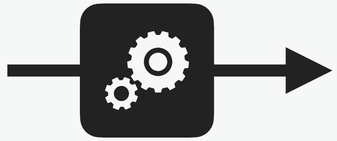
Conservative extension of MLTT with **local computation**

ROCQ APPROVED

/TheoWinterhalter/local-comp

Conclusion



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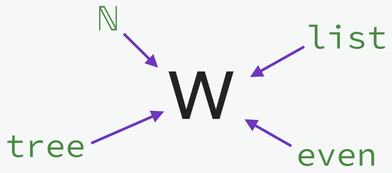

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Perspectives (ongoing)

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Concrete implementations

Agda

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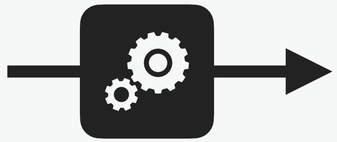

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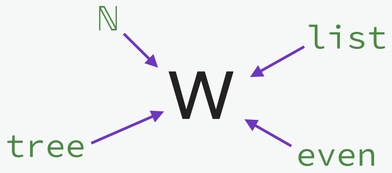

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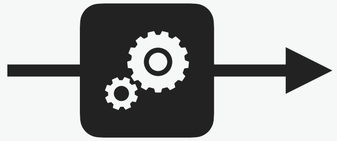

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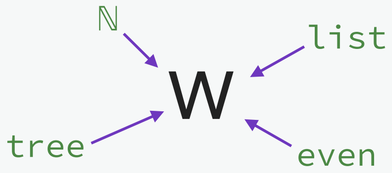

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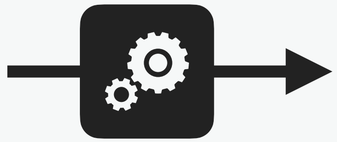

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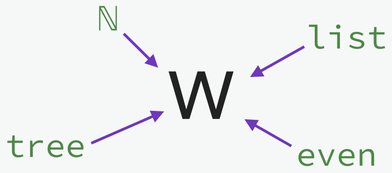

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Thank you!