TYPES 2019

Weak Type Theory

a rather strong type theory

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Computation

Conversion

Extends the notion of β-equality

$$(\lambda x.t)\ u \equiv t[x \leftarrow u]$$

Computation

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$$(\lambda x.t)\ u \equiv t[x \leftarrow u]$$

$$\mathtt{vec}_{\mathbb{N}}\ (3 + 2) \equiv \mathtt{vec}_{\mathbb{N}}\ 5$$

Computation

Conversion

Extends the notion of β-equality

$$(\lambda x.t)\ u \equiv t[x \leftarrow u]$$

$$\mathtt{vec}_{\mathbb{N}}\ (3 + 2) \equiv \mathtt{vec}_{\mathbb{N}}\ 5$$

$$[1\ ;\ 2\ ;\ 3]\ \texttt{++}\ [4\ ;\ 5] \equiv [1\ ;\ 2\ ;\ 3\ ;\ 4\ ;\ 5]$$

Computation

Witness property

from $\exists\ x,\ P\ x$

compute x

Computation

Witness property

from $\exists\ x,\ P\ x$

compute x

$$(t : \exists\ x,\ P\ x) \to \langle u ; h \rangle$$

Computation

Witness property

from ∃ x, P x

compute x

$$(t : \exists x, P\ x) \to \langle u ; h \rangle$$

with h : P u

Weak Type Theory

$$\frac{t : A \qquad A \equiv B}{t : B}$$

Weak Type Theory

Theory without computation

$$\frac{t : A \qquad A \equiv B}{t : B}$$

Weak Type Theory

Theory without computation

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$$u =_A v$$

Weak Type Theory

Theory without computation

$$u =_A v$$

$$\mathtt{refl}_A\ x : x =_A x$$

Weak Type Theory

Theory without computation

```
u =A v

reflA x : x =A x

coeA : ∀ x y, x =A y → ∀ P, P x → P y
```

Weak Type Theory

Theory without computation

$$u =_A v$$

$$\mathtt{refl}_A\ x : x =_A x$$

$$\mathtt{coe}_A : \forall\ x\ y,\ x =_A y \to \forall\ P,\ P\ x \to P\ y$$

$$\mathtt{uip}_A : \forall\ p\ q,\ p =_{x\ =\ y} q$$

Weak Type Theory

Theory without computation

$$u =_A v$$

$$\mathtt{refl}_A\ x : x =_A x$$

$$\mathtt{coe}_A : \forall\ x\ y,\ x =_A y \to \forall\ P,\ P\ x \to P\ y$$

$$\mathtt{uip}_A : \forall\ p\ q,\ p =_{x\,=\,y} q$$

$$\beta\ t\ u : (\lambda x.t)\ u = t[x := u]$$

Weak Type Theory

Theory without computation

$u =_A v$

$\mathtt{refl}_A\ x : x =_A x$

$\mathtt{coe}_A : \forall\ x\ y,\ x =_A y \to \forall\ P,\ P\ x \to P\ y$

$\mathtt{uip}_A : \forall\ p\ q,\ p =_{x = y} q$

$\beta\ t\ u : (\lambda x.t)\ u = t[x := u]$

and more...

Weak Type Theory

Congruence for free

$$\text{coe}_A : \forall\ x\ y,\ x =_A y \to \forall\ P,\ P\ x \to P\ y$$

Weak Type Theory

Congruence for free

```
coeA : ∀ x y, x =A y → ∀ P, P x → P y
```

```
x = y → y = z → x = z
```

Weak Type Theory

Congruence for free

$\mathrm{coe}_A : \forall\ x\ y,\ x =_A y \to \forall\ P,\ P\ x \to P\ y$

$x = y \to y = z \to x = z$

$f = g \to x = y \to f\ x = g\ y$

Weak Type Theory

Congruence for free ??

$\text{coe}_A : \forall\ x\ y,\ x =_A y \to \forall\ P,\ P\ x \to P\ y$

Weak Type Theory

Congruence for free ??

$\mathtt{coe}_A : \forall\, x\ y,\ x =_A y \to \forall\, P,\ P\ x \to P\ y$

Problems under binders

$$\frac{\vdash A = A' \qquad x : A \vdash B = B'}{\vdash \Pi(x{:}A).B = \Pi(x{:}A').B'}$$

Weak Type Theory

Congruence for free ??

$\mathrm{coe}_A : \forall\, x\ y,\ x =_A y \to \forall\, P,\ P\ x \to P\ y$

Problems under binders

$$\frac{\vdash A = A' \qquad x : A \vdash B = B'}{\vdash \Pi(x{:}A).B = \Pi(x{:}A').B'}$$

⇒ more axioms needed

Spectrum of Type Theories

ETT

ITT

WTT

Spectrum of Type Theories

ETT

ITT

WTT

no computation

no conversion

Spectrum of Type Theories

ETT

$$\frac{p : u = v}{u \equiv v}$$

equality reflection

ITT

WTT

no computation

no conversion

Spectrum of Type Theories

ETT $\supset$ ITT $\supset$ WTT

$$\frac{p : u = v}{u \equiv v}$$

equality reflection

no computation

no conversion

Spectrum of Type Theories

ETT

ITT

WTT

Spectrum of Type Theories

ETT

ITT

WTT

terms up to computation

Spectrum of Type Theories

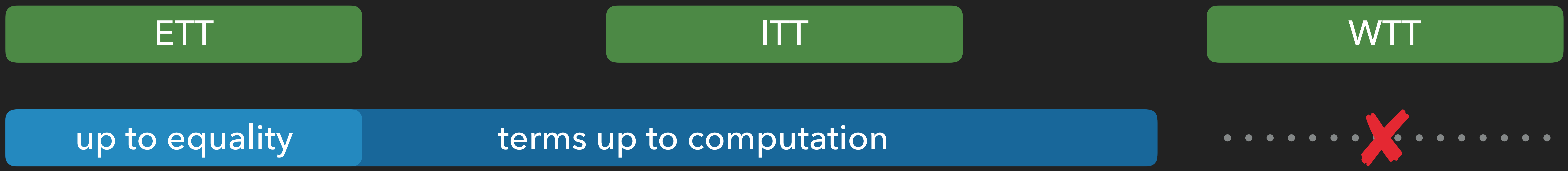
ETT

ITT

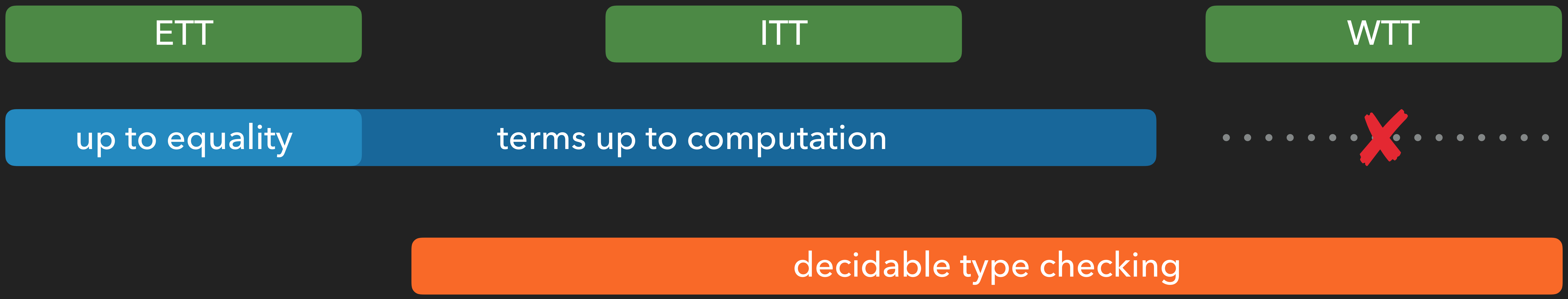
WTT

terms up to computation

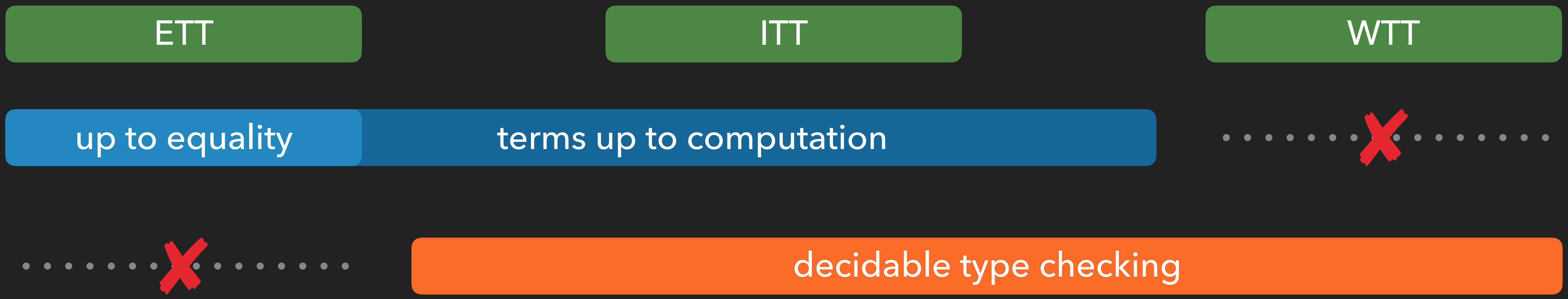
Spectrum of Type Theories



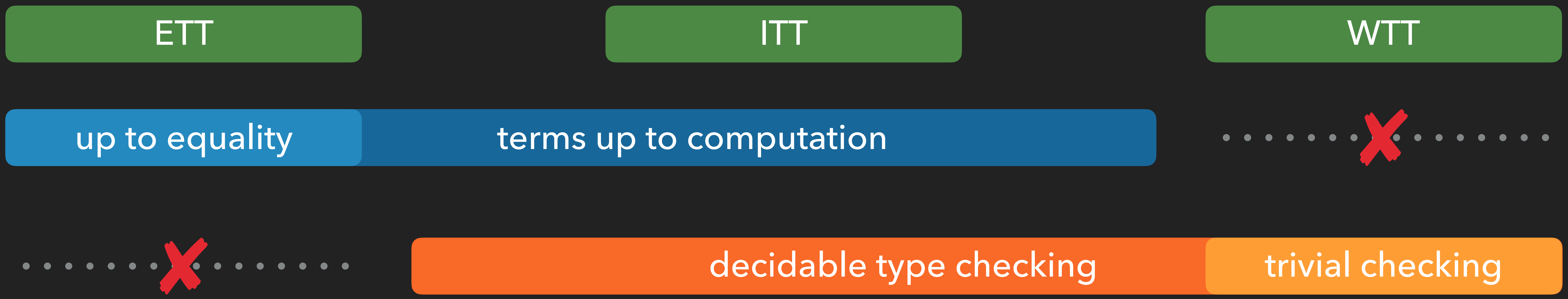
Spectrum of Type Theories



Spectrum of Type Theories



Spectrum of Type Theories



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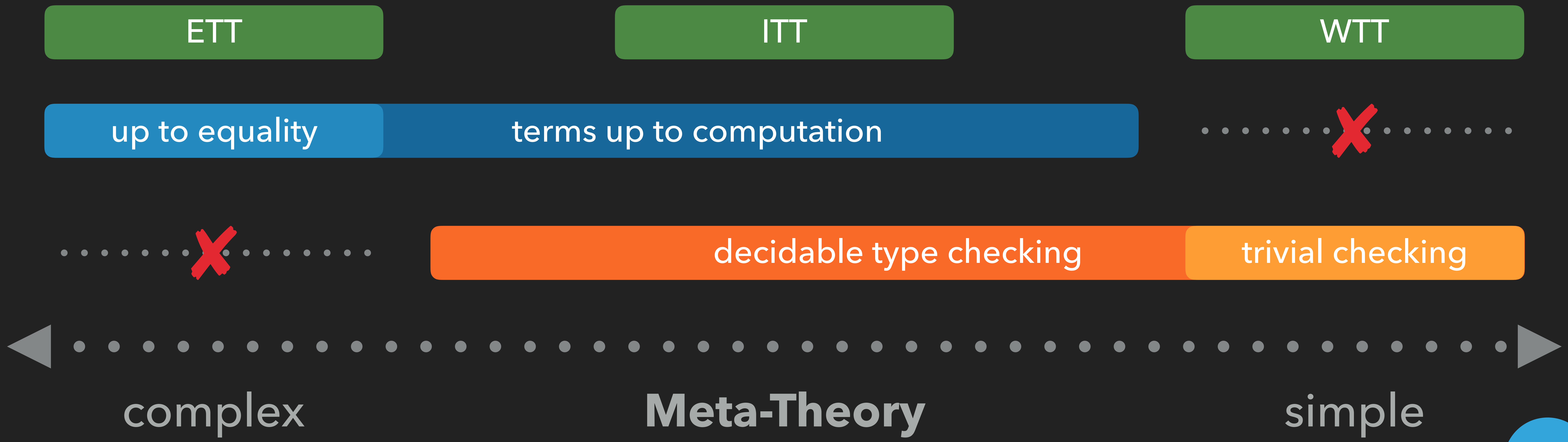
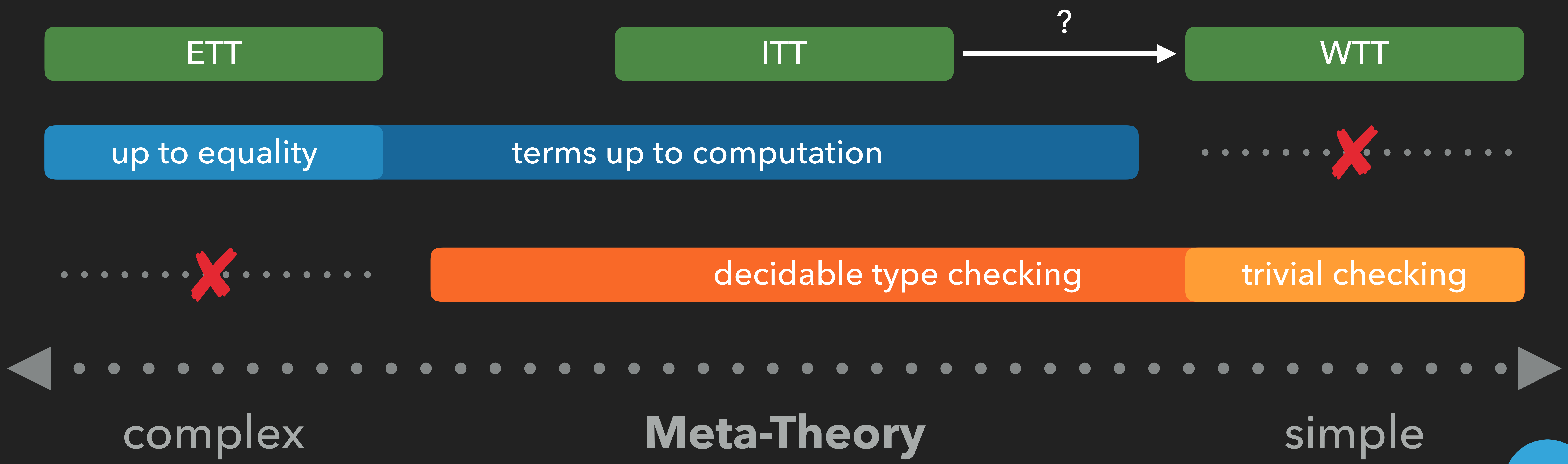
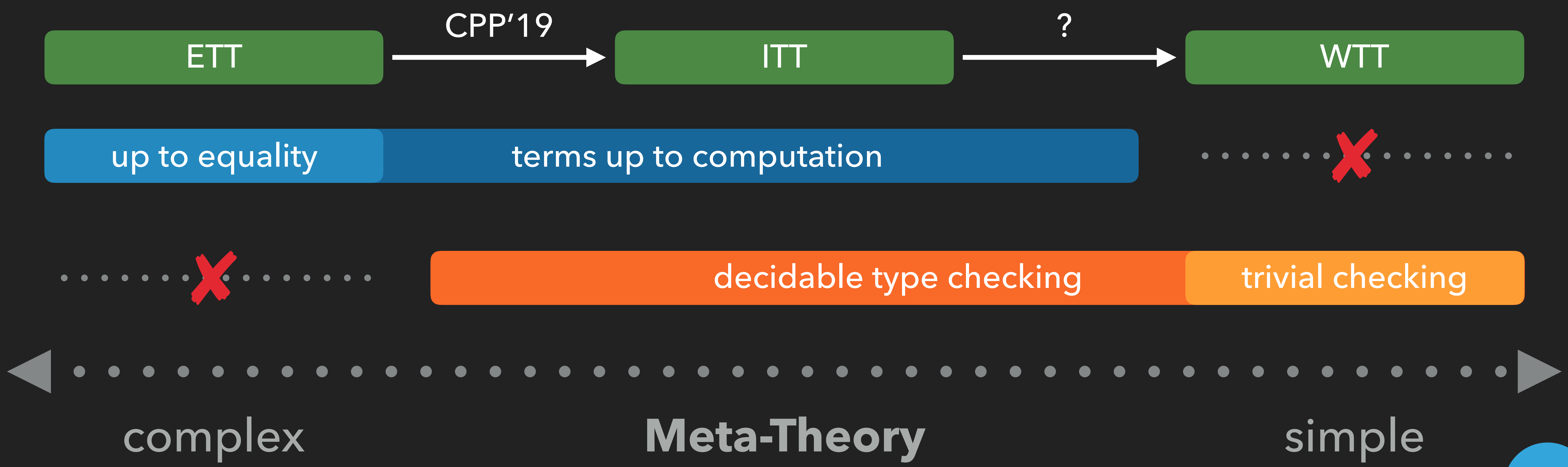


Spectrum of Type Theories



Spectrum of Type Theories



Spectrum of Type Theories